Semiclassics for a Dissipative Quantum Map

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We present a semiclassical analysis for a dissipative quantum map with an area–nonpreserving classical limit. We show that in the limit $\hbar \to 0$ the trace of an arbitrary natural power of the propagator is dominated by contributions from periodic orbits of the corresponding classical dissipative motion. We derive trace formulae of the Gutzwiller type for such quantum maps. In comparison to Tabor's formula for area–preserving maps, both classical action and stability prefactor are modified by the dissipation. We evaluate the traces explicitly in the case of a dissipative kicked top with integrable classical motion and find good agreement with numerical results.

I. INTRODUCTION

A good deal of what we know about the quantum mechanics of classically chaotic systems is due to Gutzwiller's trace formula [1,2]. It expresses the fluctuation of the density of states to lowest order in Planck's constant $\hbar$ by a sum over classical periodic orbits. Gutzwiller developed this semiclassical formalism for conservative Hamiltonian systems. Later on, the formalism was applied by Tabor to area–preserving maps [3]. A lot is known about the quantum mechanical as well as the classical dynamics of dissipative chaotic systems. While the quantum energy spectra of conservative chaotic systems tend to have spectral statistics like the well known ensembles of random Hermitian matrices [5], the spectrum of the quantum mechanical propagator of dissipative chaotic systems seems to obey the statistics of Ginibre's ensemble. This was shown numerically by analyzing a damped version of the kicked top [6]. The eigenphases become complex and the spacing distribution $P(s)$ of nearest eigenvalues shows a universal small–s behavior, $P(s) \propto s^3$, as soon as the dissipation is strong enough. For a classically integrable dissipative system, $P(s) \propto s$. This is at variance with the result for conservative systems, with the $P(s)$ of Wigner and Dyson, $P(s) \propto s^\beta$ ($s \ll 1$) with $\beta = 1, 2, 4$ depending on the time– and spin–reversal symmetry of the system. For classically integrable conservative systems, the spectrum is uncorrelated and $P(s) = \exp(-s)$ [7].
Classical dissipative and chaotic dynamics typically leads to phase–space structures dominated by strange attractors, i.e. attractors of fractal (or multifractal) dimension, whereas for conservative systems initial distributions in phase space rapidly converge towards a stationary uniform one on the energy shell [8].

Particularly noteworthy earlier semiclassical work on dissipative processes is the quantization of Henon's map with dissipation by Graham and Tél [9]; Dittrich and Graham considered the kicked rotator with dissipation [10]. In [11] a periodic–orbit analysis of the classical spectral density for the Lorenz model, the Hausdorff dimension of the Lorenz attractor, and the average Lyapunov exponent was put forward. Refs. [12] and [13] treat numerically the transition from quantum mechanics to classical mechanics for the kicked top with dissipation. In [12] it was shown by direct numerical diagonalization of the propagator of the density matrix that the eigenstate corresponding to the lowest eigenvalue when written in a coherent state basis tends towards the corresponding stationary classical distribution function, in particular to (a smoothed out version) of the strange attractor in the case of classically chaotic motion. In [13] the same analysis was carried further by solving the equivalent stochastic Schrödinger equation. This procedure allowed for much larger values of the angular momentum and correspondingly smaller values of $\hbar$ and showed again a remarkable agreement between the two stationary distributions in phase space for the quantum and classical case.

In this paper we establish a periodic-orbit theory for a dissipative quantum map P that consists of a purely unitary step followed by a purely dissipative one, the dissipative one describable by a Markovian master equation. We show that to leading order in $\hbar$ traces of arbitrary natural powers of P are given by a Gutzwiller type formula,

$$\mathrm{tr}P^N = \sum_{p.p.} \frac{\mathrm{e}^{J\sum_{i=1}^N R_i}}{\left|\mathrm{tr}\prod_{i=N}^1 M_d^{(i)} - \mathrm{tr}M\right|}, \qquad N = 1, 2, \ldots , \tag{1.1}$$

where the sum is over all periodic points of the corresponding classical map P_{cl}^N in phase space. The action appearing in the exponent is a classical action from the dissipative steps in P_{cl}^N, the matrices $M_d^{(i)}$ are the monodromy matrices for the dissipative steps and M is the total monodromy matrix of P_{cl}^N. The inverted order of the indices at the product indicates that the matrices are ordered from left to right according to decreasing indices. We will discuss (1.1) in detail below and illustrate our reasoning for a dissipative and periodically driven top, which in the integrable case allows for explicit evaluation of all traces.

By tops [14,15] we mean systems whose dynamical variables are the three components of an angular momentum, $J_{x,y,z}$, the squares of which have a conserved sum, $\mathbf{J}^2 = j(j+1)$. In the classical limit (formally attained by letting the quantum number j approach infinity) the surface of the unit sphere $\lim_{j\to\infty}(\mathbf{J}/j)^2 = 1$ becomes the phase space, such that one confronts but a single degree of freedom. Classical nonintegrability can if desired be enforced by periodic driving. Quantum mechanically, one deals with a $(2j+1)$ dimensional Hilbert space within which one must diagonalize the Floquet operator to get the spectrum of $(2j+1)$ quasienergies. The semiclassical limit is characterized by large values, integer or half integer, of the quantum number j. The convenience of periodically driven tops for investigations of spectral properties mostly lies in the compactness of the classical phase space and the corresponding finite dimensionality of the quantum mechanical Hilbert space.

Our theory applies to physical situations where dissipation is important. One might try experimental realisations with several identical two–level atoms collectively interacting with a single mode of the electromagnetic field, as in the superradiance experiment of Haroche [23].

The paper is organized as follows: In Section II we introduce the map to be studied, the composition of a purely unitary and a purely dissipative part. Semiclassical approximations for both components of the map are reviewed. Section III is devoted to the derivation of a trace formula for the propagator and Section IV deals with the generalization to higher traces. In V we apply the theory to an integrable dissipative kicked top and compare with numerical results, and in Section VI we summarize our main conclusions.

II. THE MAP

While the stroboscopic description of unitary dynamics employs a wave function ψ and a unitary Floquet operator F as $\psi(nT) = F^n\psi(0)$, a dissipative quantum map requires a density matrix $\rho(nT) = P^n\rho(0)$ with a nonunitary propagator P. For the sake of convenience we imagine dissipative and purely unitary motion to take place separately and in sequence during each period such that the map takes the form

$$\rho((n+1)T) = DF\rho(nT)F^\dagger \equiv P\rho(nT)\,; \tag{2.1}$$

in contrast to the unitary part of this map (which factorizes in F and $F^\dagger$) the dissipative part D cannot be split into two factors. Our goal is to determine spectral properties of the total propagator P, a linear, non–unitary operator, by semiclassical means. As a preparation we briefly review what is known about semiclassical approximations of the two components of the map, starting with the unitary part.

A. Semiclassics for the unitary evolution

The Floquet operator F may by itself describe a motion with either an integrable or chaotic classical limit. A simple integrable example is a pure rotation by an angle β about, say, the y-axis with $F = \exp(-\mathrm{i}\beta J_y)$. Most illustrations of general results will in fact be given for that case. If desired, a chaotic top can be treated by including a torsion about the z-axis as $F = \exp(-\mathrm{i}\frac{k}{2j+1}J_z^2)\exp(-\mathrm{i}\beta J_y)$. The corresponding classical motion proceeds in the spherical phase space defined by the conservation law $\mathbf{J}^2 = j(j+1)$ and may be described by a pair of canonical variables; a convenient choice is attained by specifying the orientation of the classical unit vector $\lim_{j\to\infty}\frac{\mathbf{J}}{j}$ by a polar angle Θ with respect to the z-axis and an azimutal angle ϕ reckoned against the x-axis in the xy-plane and then taking $\cos\theta$ as the momentum and ϕ as the conjugate coordinate.

A semiclassical approximation for F can be derived by representing F in the J_z basis ($J_z|m\rangle = m|m\rangle$). The matrix element $F_{nm} = \langle n|F|m\rangle$ is the probability amplitude to get the z-projection of the angular momentum mapped from the initial value m to the final one n. If F describes a pure rotation that matrix element becomes Wigner's d–function whose asymptotic behavior for large $J = j + \frac{1}{2}$ is well known [16]. The discrete quantum numbers m and n can be replaced in the limit $J \to \infty$ by the continuous classical initial and final momenta

$$\mu = m/J = \cos\Theta_i, \text{ and } \nu = n/J = \cos\Theta_f\,, \tag{2.2}$$

respectively. The semiclassical approximation has the familiar van Vleck form in which a classical action $S(\nu,\mu)$ enters as a phase, and its second mixed derivative

$$C(\nu,\mu) = \frac{(-1)^j}{\sqrt{2\pi J}}\sqrt{|\partial_\nu\partial_\mu S(\nu,\mu)|} \tag{2.3}$$

as a preexponential factor. All classical orbits connecting the initial and the final momentum contribute a term of the structure $C(\nu,\mu)\mathrm{e}^{\mathrm{i}JS(\nu,\mu)}$. Explicit expressions for the function $S(\nu,\mu)$ can be found in [18], both for integrable cases such as pure rotations and chaotic ones involving rotations and torsions. For our present prime example, the linear rotation by β about the y-axis, there are typically two classical trajectories differing by the signs of azimuth and action; only if $\beta > \theta_i + \theta_f$ or $\beta < |\theta_i - \theta_f|$, no classical trajectories connecting the initial with the final momentum exist; if one of the foregoing inequalities is turned into an equality we have $\phi_i = \phi_f = 0$ and only one trajectory contributes (see the discussion in [18]); we shall not consider these latter exceptions here but only treat the case of two contributing trajectories for which the amplitude under discussion reads

$$F_{nm} \simeq \sum_{\sigma=\pm 1} C(\nu,\mu)\mathrm{e}^{\mathrm{i}\sigma(JS(\nu,\mu)-\frac{\pi}{4})}\,. \tag{2.4}$$

The action $S(\nu,\mu)$ (or $-S$ if $\phi < 0$) also serves as the generating function of the classical map

$$(\mu,\phi_i) \to (\nu,\phi_f) \tag{2.5}$$

in the sense

$$\partial_\nu S(\nu,\mu) = \mp\phi_f(\nu,\mu),\ \partial_\mu S(\nu,\mu) = \pm\phi_i(\nu,\mu)\,; \tag{2.6}$$

here, the upper (lower) sign refers to $\phi_{i,f} > 0$ (< 0); note that for the rotation about the y–axis the sign of the initial and final azimuth is the same.

The generating property (2.6) and the general van Vleck form (2.4) will allow us to develop semiclassical formulas largely independent of the particular dynamics. In particular, the specific form of $S(\nu,\mu)$ will not be needed below.

B. Semiclassics for the dissipative part of the map

To introduce the dissipative part D of our map (2.1) we momentarily allow for a continuous time and employ the master equation of spin damping well known from the theory of superradiance [22–24],

$$\dot{\rho(t)} = \kappa([J_-,\rho(t)J_+] + [J_-\rho(t),J_+]) \equiv \Lambda\rho(t)\,, \tag{2.7}$$

where the damping constant κ sets the time scale on which the angular momentum in question approaches, while keeping its square $\mathbf{J}^2 = j(j+1)$ conserved, the stationary state of minimal z-component, $J_z = -j$; the operators $J_\pm = J_x \pm \mathrm{i}J_y$ are the familiar raising and lowering operators.

Let us first look at the *classical* limit. The classical equations of motion can be determined by extracting equations for the expectation values of J_z and $J_\pm$, parameterizing them as $\langle J_z\rangle = J\cos\Theta$ and $\langle J_\pm\rangle = J\sin\theta\mathrm{e}^{\pm\mathrm{i}\phi}$ and factorizing all operator products (e.g. $\langle J_+J_-\rangle \to \langle J_+\rangle\langle J_-\rangle$). One then finds that the angular momentum behaves classically like an overdamped pendulum [15],

$$\dot{\phi} = 0, \qquad \dot{\Theta} = 2J\kappa\sin\Theta\,. \tag{2.8}$$

The classical damping constant is thus revealed as $2J\kappa$. For future convenience, we give the solution of (2.8) in terms of the dimensionless time $s = 2J\kappa t$ and phase–space variables $\mu(s) = \cos\Theta(s)$ and ϕ:

$$s = \frac{1}{2}\ln\frac{(1-\mu(s))(1+\mu(0))}{(1-\mu(0))(1+\mu(s))}, \qquad \phi(s) = \phi(0)\,. \tag{2.9}$$

For the sake of definiteness we imagine the damping to be effective during the full duration T of a period, thus confining the previous unitary motion to an instantaneous kick. The dissipative part of the map is then obtained by exponentiating the continuous-time generator Λ of the above master equation as $D = \exp(\Lambda T)$.

The *semiclassical* approximation for the dissipative propagator $D = \exp(\Lambda T)$ looks much like the familiar van Vleck form for unitary propagators. Details of the theory which amounts to solving a Hamilton–Jacobi equation for the classical generating function of the dissipative motion can be found in [21]. Here we limit ourselves to exposing the idea and summarizing the main results.

Looking at the master equation (2.7) in the J_z basis we reveal an important conservation law upon expressing the density matrix element $\langle m_1|\rho(s)|m_2\rangle$ as a function of the mean $m = \frac{m_1+m_2}{2}$ and the skewness $k = \frac{m_1-m_2}{2}$ of

the quantum numbers m_1, m_2, i.e. $\rho_m(k,s) = \langle m_1|\rho(s)|m_2\rangle$. The operators $J_\pm$ give rise to the functions $d_m = j(j+1) - m(m-1)$ known from the angular momentum algebra ($J_\pm|m\rangle = \sqrt{d_{\mp m}}|m\pm 1\rangle$) and $f_m(k) = \sqrt{d_{m+k}d_{m-k}}$. With these abbreviations our master equation reads

$$J\frac{d\rho_m(k,s)}{ds} = f_{m+1}(k)\rho_{m+1}(k,s) - (d_m - k^2)\rho_m(k,s)\,. \tag{2.10}$$

Obviously, the skewness k enters only as a parameter and is conserved. The general solution of (2.10) can be written with the help of a "dissipative propagator" $D_{mn}(k,s)$ as

$$\rho_m(k,s) = \sum_n D_{mn}(k,s)\rho_n(k,0)\,. \tag{2.11}$$

For large J, the variables $\mu = m/J$ and $\nu = n/J$ prove useful again. A third one is introduced for k, $\eta = k/J$, and we write consequently $D_{mn}(k,s) = J^{-1}D(\mu,\nu;\eta;s)$.
The clue towards D is a WKB type ansatz,

$$D(\mu,\nu;\eta;s) = B(\mu,\nu;\eta;s)\exp(JR(\mu,\nu;\eta;s))\,, \tag{2.12}$$

and to expand systematically in $1/J$. Note that even though the functions B and R are assumed smooth, the WKB form still allows for a delta peak as $J\to\infty$. To lowest order in $1/J$, (2.11) leads to an equation for R which looks like a Hamilton–Jacobi equation for a one dimensional Hamiltonian system with the somewhat unconventional Hamilton function $H(\mu,p) = (1-\mu^2)\,(1-\exp(p))$,

$$\frac{\partial R}{\partial s} + H(\mu, \frac{\partial R}{\partial \mu}) = 0\,. \tag{2.13}$$

In this Hamiltonian description of the semiclassical approximation of our damped spin the variable μ appears as a coordinate with a canonically associated momentum $p = \frac{\partial R}{\partial \mu}$. The Hamilto–Jacobi equation (2.13) is equivalent to two Hamilton equations of motion for μ and p. The solutions involve two integration constants, one of which we choose as the initial coordinate ν; as the second will serve the conserved "energy" $E = H(\mu,p)$ which for the sake of convenience we represent by the parameter $a = \sqrt{1-E}$. With this choice, we get the Hamiltonian trajectories

$$s = \frac{1}{2a}\ln\frac{(\nu+a)(\mu-a)}{(\nu-a)(\mu+a)}\,. \tag{2.14}$$

Comparison with the classical dynamics (2.9) shows that μ and ν are connected by a classical trajectory iff $a = 1$. If one thinks of a Hamiltonian trajectory as specified by the initial and final coordinates ν and μ with a given time span s, as we shall frequently do below, one should read (2.14) as determining the parameter a as a function $a = a(\mu,\nu,s)$. The action R can be obtained by integrating $pd\mu$ along the trajectory (2.14) and reads [21]

$$\begin{aligned} R(\mu,\nu;\eta;s) = \frac{1}{2}(&\xi(1,\nu-\eta) - \xi(1,\mu-\eta) + \xi(1,\nu+\eta) - \xi(1,\mu+\eta)) \\ &-\xi(a,\nu) + \xi(a,\mu) + s(a^2 - 1 + \eta^2)\,, \\ \xi(x,y) \equiv\; &(x+y)\ln(x+y) - (x-y)\ln(x-y)\,. \end{aligned} \tag{2.15}$$

Several features of the action R will be of importance in the sequel:

- For $\eta = 0$ we get $a = 1$ from $\partial_\mu R = 0$, thus the classical equation of motion (2.9); then $\partial_\nu R = 0$ holds as well.
- For $\eta = 0$ and μ, ν connected by the classical trajectory (i.e. $a = 1$), R is strictly zero. This actually holds beyond the semiclassical approximation, as it can be traced back to conservation of probability by the master equation for $k = 0$ [20].
- R is an even function of η and has in fact always a *maximum* at $\eta = 0$.

An equation like (2.13) has already been found for a dissipative problem in [19], but the preexponential factor which is crucial to our problem was not treated there. That prefactor is obtained in next-to-leading order in $1/J$ as [21]

$$B(\mu,\nu;\eta;s) = \sqrt{\frac{J}{2\pi}}\sqrt{\left.\frac{\partial\nu}{\partial\mu}\right|_a}\sqrt{\partial_\mu\partial_\nu R(\mu,\nu;\eta;s)}\,; \tag{2.16}$$

it differs from the familiar form $\sqrt{\partial_\mu \partial_\nu S(\nu,\mu)}$ in the van Vleck formula for unitary propagators by the additional factor $\sqrt{\frac{\partial \nu}{\partial \mu}\mid_a}$. Since $\sqrt{\partial_\mu \partial_\nu S(\nu,\mu)}$ arises from the propagation of a wave–function we might expect the square of this factor when propagating a density matrix with $\eta = 0$ (i.e. probabilities). And indeed, both square roots in (2.16) are the same in the semiclassical limit and combine to the classical Jacobi determinant: For $J \to \infty$ we can restrict our attention to values of μ and ν close to the classical trajectory, i.e. $a = 1$, as the propagator for other values is exponentially small. We call this trajectory $\mu_d(\nu)$ where the time is fixed. The first square root just gives $\sqrt{\frac{d\mu_d^{-1}(\mu)}{d\mu}}$. On the other hand, since $R(\mu,\nu;0;s)$ has a maximum on the classical trajectory, it can be written as $R(\mu,\nu;0;s) \simeq -\frac{\alpha}{2}(\nu - \mu_d^{-1}(\mu))^2$ close to the maximum with some parameter $\alpha > 0$. This means $\partial_\mu \partial_\nu R(\mu,\nu;0;s) = \alpha \frac{d\mu_d^{-1}}{d\mu}$, and for $J \to \infty$ (2.16) therefore leads to the classical propagator of *probabilities*,

$$D_{cl}(\mu,\nu,\ 0;s) = \left|\frac{d\mu_d^{-1}}{d\mu}\right| \delta(\nu - \mu_d^{-1}(\mu))\,.$$

As a conclusion, we may write the preexponential factor for $\eta = 0$ as

$$B(\mu,\nu;0;s) \simeq \sqrt{\frac{J}{2\pi}} \sqrt{\left|\frac{d\mu_d^{-1}(\mu)}{d\mu}\right|} \sqrt{\partial_\mu \partial_\nu R(\mu,\nu;0;s)}\,, \tag{2.17}$$

if it is evaluated on the classical trajectory. This is what we will need for the semiclassical evaluation of the traces of P (see the next section).

A word is in order here about the limits of validity of the present semiclassical treatment of dissipation. It disregards the existence of discrete energy levels and becomes unacceptable when this discreteness is important. These are the cases when

- the initial state n and the final state m coincide or are separated by just a few intermediate levels: $n - m \sim 1$;
- m or n (or both) coincide with the highest or lowest energy state of the system $\mid \pm j\rangle$ or are separated from them by just a few levels, i.e. when at least one of the following relations hold $\pm j \pm n \sim 1, \pm j \pm m \sim 1$.

The first limitation implicitly puts a lower bound on the time, $s \gg \mathcal{O}(J^{-1})$, because for small propagation times it is only the elements D_{mn} with $n - m$ zero or of order unity which are noticeably different from zero. The failure of the continuum approximation is self evident in this case: the probability literally had no time to get away from the initial level to form a smooth distribution. The second limitation means that the initial and final coordinates should not be too close to the "poles" $\pm j$. The inapplicability of the naive semiclassical approximation to describe the lowest energy state of a quantum system is well-known; in systems with the spectrum bounded from above this is also true about the highest-energy state. Again, there is an implicit limitation on the time which should be not too large (small compared with J) so that the probability is not concentrated close to the ground state.

In the present work the deficiencies of the semiclassical approximation just mentioned seem to be of no importance, at least for the lowest traces of P. For higher traces we see deviations to next order in $1/J$ which may well be due to our neglecting the discreteness of the spectrum. A better approximation for the dissipative propagator exists which allows for uniform evaluation, at least for not very large times [21].

Since both F and D have a classical counterpart, so does the combined map P. We shall denote it as P_{cl} and propose to show that its periodic points determine the spectral properties of P.

III. SEMICLASSICAL EVALUATION OF THE FIRST TRACE OF P

We first write out an exact formal expression for the quantum map P in terms of the discrete quantum numbers. Immediately after the unitary motion, ρ has the elements $\rho_n(k,0+) = \langle n_1|F\rho(0)F^\dagger|n_2\rangle = \sum_{l_1,l_2=-j}^{j} F_{n_1 l_1}(F^\dagger)_{l_2 n_2}\langle l_1|\rho(0)|l_2\rangle = \sum_{l,r} F_{n+k,l+r}F^*_{n-k,l-r}\rho_l(r,0)$ with $l = \frac{l_1+l_2}{2}$ and $r = \frac{l_1-l_2}{2}$. We insert this in the general solution (2.11) and get $\rho_m(k,\tau) = \sum_{n,l,r} D_{mn}(k,\tau)F_{n+k,l+r}F^*_{n-k,l-r}\rho_l(r,0) = \sum_{l,r} P_{mk;lr}\rho_l(r,0)$ where we have introduced the dimensionless parameter $\tau = 2J\kappa T$. The propagator P is read off as

$$P_{mk;lr} = \sum_n D_{mn}(k,\tau)F_{n+k,l+r}F^*_{n-k,l-r}, \tag{3.1}$$

and yields the first trace

$$\mathrm{tr}P = \sum_{m,k} P_{mk;mk} = \sum_{m,n,k} D_{mn}(k,\tau) F_{n+k,m+k} F^*_{n-k,m-k} \,. \tag{3.2}$$

We now replace sums by integrals via Poisson summation, employ the semiclassical propagators derived in the previous section, and shall eventually integrate by the saddle–point method. The integrals to be done read

$$\begin{aligned}\mathrm{tr}P = J^2 \sum_{r,l,t=-\infty}^{\infty} \int 2d\mu d\nu d\eta \exp[J\Phi(\mu,\nu;\eta)] \sum_{\sigma_1,\sigma_2} \mathrm{e}^{\mathrm{i}\frac{\pi}{4}(\sigma_1+\sigma_2)} \\ \cdot B(\mu,\nu,\eta) C(\nu+\eta,\mu+\eta) C^*(\nu-\eta,\mu-\eta)\,. \end{aligned} \tag{3.3}$$

They involve the now complex action

$$\begin{aligned}\Phi(\mu,\nu;\eta) &= R(\mu,\nu;\eta) + \mathrm{i}2\pi(r(\mu+\eta) + l(\nu+\eta) + 2t\eta) \\ &\quad + \mathrm{i}\left\{\sigma_1 S(\nu+\eta,\mu+\eta) + \sigma_2 S(\nu-\eta,\mu-\eta)\right\} \\ &= \Phi_0(\mu,\nu;\eta) + \mathrm{i}\Phi_1(\mu,\nu;\eta)\end{aligned} \tag{3.4}$$

whose real part is related to dissipation while the imaginary part represents unitary dynamics (We apologize to the reader for frivolously calling action what is usually called $-\mathrm{i}\times$ action). The integral contains factors 2 in front of the integer t and before $d\mu$ which arise because m, n and k can be simultaneously half–integer. For brevity, we have suppressed the parameters β and τ.

Three complex saddle–point equations (SPE) have to be fullfilled: $\partial_\mu\Phi = 0$, $\partial_\nu\Phi = 0$ and $\partial_\eta\Phi = 0$. We look for real solutions (complexity would indeed look unphysical, as $m = \mu J \in \mathcal{Z}$ etc.). Formally we can not exclude the existence of further complex solutions, but as long as classical solutions exist we expect them to dominate over non–classical ones. This is certainly true in the case of the superradiance dissipation where $R = 0$ on the classical trajectory, whereas complex solutions would lead to exponential supression from the term $i\Phi_1$. But even more generally we expect classical solutions to dominate, since they are known to dominate in non–dissipative quantum mechanics and dissipation favors classical behavior even more.

We thus need separately $\partial_\mu\Phi_0 = 0$ and $\partial_\mu\Phi_1 = 0$, where Φ_0 and $\mathrm{i}\Phi_1$ are the real and imaginary parts of Φ (and correspondingly for the derivatives ∂_ν and ∂_η). These are six equations for the three variables μ, ν, η, but the additional three integers r, l, t from the Poisson summation do allow for a solution.

The first equation, $\partial_\eta\Phi_0 = 0$, immediately gives $\eta = 0$ with the important consequence that only the propagation of probabilities contributes to the trace in the semiclassical limit. Inserting $\eta = 0$ in $\partial_\eta\Phi_1 = 0$, we are led to $(\sigma_1 - \sigma_2)(\partial_\nu + \partial_\mu)S(\nu,\mu) + 2\pi(r + l + 2t) = 0$.

Consider first the case $\sigma_1 = -\sigma_2$. In view of the generating properties of $S(\nu,\mu)$,the foregoing equation yields $\phi_f(\nu,\mu) = \phi_i(\nu,\mu) + \pi(r + l + 2t)$. From $\partial_\mu\Phi_1 = 0 = 2\pi r + (\sigma_1 + \sigma_2)\partial_\mu S(\nu,\mu) = 0$ we conclude $r = 0$; and from $\partial_\nu\Phi_1 = 0$ correspondingly $l = 0$. We thus obtain the unique classical trajectory for the unitary evolution which connects the initial and final momenta for the given time span τ, uniqueness due to the additional requirement that initial and final azimuths be the same up to a multiple of 2π! The second of the three saddle-point equations for the real part, $\partial_\nu\Phi_0 = 0$, immediately leads to $a = 1$ (since $\eta = 0$), i.e. precicely the condition distinguishing the trajectories of the classical overdamped pendulum within the larger family of Hamiltonian tracjectories describing the semiclassical limit of the dissipative motion. With that condition met, the last equation $0 = \partial_\mu\Phi_0 = -\partial_\nu\Phi_0$ also holds without further restriction.

To summarize, the saddle-point equations determine the classical trajectory which leads, in the unitary part of the map, from μ to ν with constant ϕ and, for the dissipative part, back from ν to μ, again under constant ϕ. The saddle points of the above integrand are the *fixed points of the classical map.*

Consider now $\sigma_1 = \sigma_2 \equiv \sigma$. The imaginary part of the η variation, $\partial_\eta\Phi_1 = 0$, does not restrict ν, μ, η but just gives $r + l + 2t = 0$. New information now comes from $\partial_\nu\Phi_1 = 0$ and $\partial_\mu\Phi_2 = 0$ and amounts to $\phi_f = \pi l = -\pi(r + 2t)$ and $\phi_i = -\pi r$. So we have again $\phi_f = \phi_i$ up to an integer multiple of 2π, but additionally $\phi_i = 0$ or π. The dissipative part remains unchanged, such that for the combined map we get the particular class of periodic orbits that take place at $\phi = 0$ or π. For our integrable dissipative kicked top, such orbits are only possible if the damping constant τ exceeds a certain critical value τ_c. Moreover, we shall see that these orbits lie exactly on the boundary between classically allowed and classically forbidden motion concerning the rotation alone (cf. secII A). The semiclassical approximation (2.4) then ceases to be valid and should be replaced by a uniform approximation bridging the classically allowed and forbidden domains [18]. This program is beyond the scope of the present work; we will restrict ourselves to regimes where this particular kind of orbits does not exist.

Note that the combination $\sigma_1 = -\sigma_2$ would arise automatically, if the van Vleck propagator contained only one classical path. The opposite sign results from the time reversed propagator $F^\dagger$. It means the interference of a given

orbit with itself.

The fact that only the interference of an orbit with itself contributes can be seen more generally in the case where the unitary part contains several classical paths with actions $S_l(\nu,\mu)$. One encounters then a term $\mathrm{i}J(S_l(\nu+\eta,\mu+\eta)-S_k(\nu-\eta,\mu-\eta))$ and a double sum over paths l,k. However, the SPEs lead to an equality of initial and final angles (modulo 2π) for the two paths l and k. Since to a given initial phase–space point there is only one classical path, only $l=k$ can contribute. Even the case of l and k designating two branches of a separatrix in phase space can in general be excluded, due to the equality also of the final phase–space point. The only exception are degenerate paths by which we mean that two originally separate paths conincide when a system parameter is suitably changed. Exactly this extraordinary situation occurs in the above example for the kicked top: For $\tau<\tau_c$ there are always two symmetry related classical paths for the unitary part (parametrized by $\sigma=\pm$), one starting at an azimuth ϕ_i, the other at an azimuth $-\phi_i$. When τ is increased both conincide for $\tau=\tau_c$ at $\phi_i=0$ and remain there for all values $\tau>\tau_c$. But at the same time classical dynamics breaks down and the van Vleck propagator ceases to be valid, anyway.

To exploit the SPA, we have to expand Φ to second order in the deviation $\mathbf{x}^T=(\delta\mu,\delta\nu,\delta\eta)$ from the fixed point $(\mu_0,\nu_0;0)$. We obtain a quadratic form with a complex symmetric 3×3 matrix $Q_{3\times3}$ that contains the second partial derivatives $\partial_{x_i}\partial_{x_j}\Phi(\mu,\nu;\eta)\mid_{\mu_0,\nu_0;0}$ in an obvious way,

$$\Phi(\mu,\nu;\eta)\simeq\Phi(\mu_0,\nu_0;0)-\frac{1}{2}\mathbf{x}^TQ_{3\times3}\mathbf{x}\,. \tag{3.5}$$

We have chosen to pull out a minus sign from the partial derivatives. The fact that $Q_{3\times3}$ is complex and therefore non–Hermitian makes the SPA non–trivial. In view of the repeated application of the saddle–point method for the integration of a complex function of several variables, we devote appendix A to a discussion of the general formula derived in [26,27].

Let us come back to our saddle-point integral (3.3). According to equation (A1) we have to evaluate $\det Q_{3\times3}$. It is easy to see that the 2×2 (μ,ν) block and the (η,η) block of the imaginary part of $Q_{3\times3}$ are zero. In addition, $\partial_\mu\partial_\eta\Phi_0=0=\partial_\nu\partial_\eta\Phi_0$, since Φ_0 has a maximum at $\eta=0$ for all μ and ν, i.e. it must have the structure $\Phi_0\simeq f_0(\mu,\nu)\eta^2$ for small η with some function f_0. The second derivatives of the real part can be simplified on the classical periodic orbit: Since $\partial_\mu\Phi_0(\mu,\nu;0)\mid_{\mu=\mu_d(\nu)}=0$ for *all* ν, we find $\frac{d}{d\nu}\partial_\mu\Phi_0(\mu,\nu;0)|_{\mu=\mu_d(\nu)}=\partial_\mu^2\Phi_0\frac{d\mu_d}{d\nu}+\partial_\mu\partial_\nu\Phi_0=0$, i.e. $\partial_\mu\partial_\nu\Phi_0=-\partial_\mu^2\Phi_0\frac{d\mu_d}{d\nu}$. Similarly, starting from $\partial_\nu\Phi_0\mid_{\mu=\mu_d(\nu)}=0$, one finds $\partial_\nu^2\Phi_0=\partial_\mu^2\Phi_0\left(\frac{d\mu_d}{d\nu}\right)^2$. We introduce the functions $h(\nu,\mu)=2(\partial_\nu+\partial_\mu)S(\nu,\mu)=\partial_\eta\Phi_1$ and $\gamma=\frac{d\mu_d}{d\nu}$ to write $Q_{3\times3}$ in the form

$$Q_{3\times3}=-\begin{pmatrix}\partial_\mu^2\Phi_0 & -\gamma\partial_\mu^2\Phi_0 & \mathrm{i}\partial_\mu h\\ -\gamma\partial_\mu^2\Phi_0 & \gamma^2\partial_\mu^2\Phi_0 & \mathrm{i}\partial_\nu h\\ \mathrm{i}\partial_\mu h & \mathrm{i}\partial_\nu h & \partial_\eta^2\Phi_0\end{pmatrix}. \tag{3.6}$$

It has the determinant

$$\det Q_{3\times3}=-\partial_\mu^2\Phi_0(\partial_\nu h+\gamma\partial_\mu h)^2\,. \tag{3.7}$$

Since at the saddle point $\partial_\mu^2\Phi_0<0$ (maximum of the propagator on the classical trajectory!), we have $\det Q_{3\times3}>0$ and by the same reason the first minor $D_1=-\partial_\mu^2\Phi_0>0$. The second upper left minor gives zero, but we can fix this by introducing a small positive ϵ in the first element. This leads to $D_2=-\epsilon\gamma^2\partial_\mu^2\Phi_0>0$. Thus, all minors are real and positive so that $\arg\det Q_{3\times3}$ is now uniquely fixed to zero (cf. (A3)).

The preexponential factor in D simplifies on the classical trajectory (cf. (2.17)): $\sqrt{\left|\frac{d\mu_d^{-1}(\mu)}{d\mu}\right|}\sqrt{\partial_\mu\partial_\nu R(\mu,\nu;0)}=\sqrt{\frac{d\mu_d^{-1}(\mu)}{d\mu}}\sqrt{-\frac{d\mu_d}{d\nu}\partial_\mu^2\Phi_0}=\sqrt{-\partial_\mu^2\Phi_0}$. From the action in the exponent only the real part $R=R(\mu,\nu;0)$ remains. We arrive at the saddle–point approximation for $\mathrm{tr}P$,

$$\mathrm{tr}P=\sum_{p.o.}\left|\frac{\partial_\mu\partial_\nu S(\nu,\mu)}{(\partial_\nu^2+\partial_\mu\partial_\nu)S(\nu,\mu)+\frac{d\mu_d}{d\nu}(\partial_\mu^2+\partial_\mu\partial_\nu)S(\nu,\mu)}\right|\mathrm{e}^{JR}\,. \tag{3.8}$$

The sum is over periodic orbits of the classical map and it is understood that all quantities are evaluated on these. The foregoing trace formula can be simplified further if we go over to phase–space coordinates μ and $\phi=\phi_i$ and exploit

the generating properties of $S(\nu,\mu)$. Let us call $\nu_r(\mu,\phi)$ the momentum component of the classical trajectory for the unitary motion (the index $_r$ stands for rotation), if the initial values μ and ϕ are given. The corresponding angular coordinate will be denoted by $\phi_r(\mu,\phi)$; it is related to $\phi_f(\nu,\mu)$ by $\phi_f(\nu,\mu)=\phi_r(\mu,\phi_i(\nu,\mu))$. The corresponding quantities for the dissipative motion are $\mu_d(\nu,\phi)=\mu_d(\nu)$ and $\phi_d(\nu,\phi)=\phi$. The generating properties of $S(\nu,\mu)$ (cf. (2.6)) imply $\frac{\partial^2_\mu S(\nu,\mu)}{\partial_\mu\partial_\nu S(\nu,\mu)}=\frac{\partial_\mu\phi_i(\nu,\mu)}{\partial_\nu\phi_i(\nu,\mu)}=-\frac{\partial\nu_r(\mu,\phi)}{\partial\mu}\mid_{\phi=\phi_i}$ and $\frac{\partial^2_\nu S(\nu,\mu)}{\partial_\mu\partial_\nu S(\nu,\mu)}=-\frac{\partial_\nu\phi_f(\nu,\mu)}{\partial_\nu\phi_i(\nu,\mu)}=-\frac{\partial\phi_r(\mu,\phi)}{\partial\phi}$. Our trace thus reads

$$\mathrm{tr}P=\sum_{p.o.}\frac{\mathrm{e}^{JR}}{\left|1+\frac{d\mu_d}{d\nu}-\frac{\partial\phi_r(\mu,\phi)}{\partial\phi}-\frac{d\mu_d}{d\nu}\frac{\partial\nu_r(\mu,\phi)}{\partial_\mu}\right|}\,. \tag{3.9}$$

Finally, the preexponential factor can be written in terms of traces of classical monodromy matrices. The monodromy matrices for the dissipative and the unitary motion are

$$M_d=\begin{pmatrix}\left(\frac{\partial\mu_d}{\partial\nu}\right)_\phi & 0\\ 0 & \left(\frac{\partial\phi_d}{\partial\phi}\right)_\mu\end{pmatrix},\quad M_r=\begin{pmatrix}\left(\frac{\partial\nu_r}{\partial\mu}\right)_\phi & \left(\frac{\partial\nu_r}{\partial\phi}\right)_\mu\\ \left(\frac{\partial\phi_r}{\partial\mu}\right)_\phi & \left(\frac{\partial\phi_r}{\partial\phi}\right)_\mu\end{pmatrix}. \tag{3.10}$$

respectively. Their product $M=M_dM_r$ is the monodromy matrix for the combined map. Recall that the dissipation conserves ϕ, such that $\frac{d\mu_d}{d\nu}=\left(\frac{\partial\mu_d}{\partial\nu}\right)\mid_\phi$. We therefore arrive at

$$\mathrm{tr}P=\sum_{p.o.}\frac{\mathrm{e}^{JR}}{|\mathrm{tr}M_d-\mathrm{tr}M|}\,. \tag{3.11}$$

This is the first central result. It generalizes Tabor's formula for classically area–preserving maps to an area–nonpreserving map and shows that even in the case of dissipative quantum maps, information about the spectrum is encoded in classical periodic orbits. All quantities must be evaluated on the periodic orbits and with $\eta=0$. We remark that the formula holds for both chaotic or integrable maps, as long as the periodic orbits are sufficiently well separated in phase space such that the SPA is valid.

For comparing (3.11) with Tabor's result, $\mathrm{tr}F=\sum_{p.o.}\frac{\mathrm{e}^{\mathrm{i}(JS+\alpha)}}{|2-\mathrm{tr}M|^{1/2}}$, one should remember that we consider the propagator of the density matrix, but Tabor the propagator of the wave function [3]. In the limit of zero dissipation, we should get $\mathrm{tr}P=|\mathrm{tr}F|^2$. That limit can unfortunately not be taken in (3.11), since our semiclassical dissipative propagator is only valid for $\tau\gtrsim 1/J$. However, $|\mathrm{tr}F|^2$ for $\tau\to 0$ would definitely lead to a double sum over periodic orbits. Of the double sum only a single sum remains in (3.11); all cross terms between different orbits are killed by dissipation (this is due to $\eta=0$!). An important quantum–mechanical effect of the dissipation is indeed the very rapid destruction of interferences on time scales much shorter than the classical time scales [15]. *Decoherence leads in the trace automatically to the "diagonal approximation"* which suppresses interferences of different orbits!

Tabor's preexponential factor is reproduced in the limit $\tau\to 0$, since M_d becomes the unit matrix, and thus $\mathrm{tr}M_d=2$ and $\mathrm{tr}M=\mathrm{tr}(M_dM_r)=\mathrm{tr}M_r$. It is, however, raised to the power 1 instead of 1/2, since we propagate a density matrix and not a wave function. The action itself is purely real; the imaginary parts from the unitary motion cancel each other at $\eta=0$. This property holds for all dissipative processes describable by a Markovian master equation that leads to a dissipative propagator with a single maximum (as a function of η) at $\eta=0$. As outlined in the section on the dissipative part, the saddle-point action is zero in our case of spin damping. Then all physics is in the preexponential factor, which describes the stability of each orbit! If $R\neq 0$, only a single periodic orbit may dominate the trace in the limit of $J\to\infty$, namely the one with the maximum value of R.

Before we calculate higher traces, let us comment about the generality of our result. In deriving (3.11) we made use of the general van Vleck forms of the propagators and the generating properties of the actions appearing in them. Within the class of dissipative processes considered the only ingredients particular to the specific problem at hand were the fact that the dissipation conserves ϕ and the sum over two paths in (2.4). However, since the dissipative processes considered impliy only interference of each orbit with itself, the latter feature does not restrict the generality of (3.11). The formula should hold independently of the number of classical paths that contribute to the unitary motion. The vanishing of the Maslov phase relies only on properties of the dissipative part and will always hold if the dissipative propagator has a maximum on the classical trajectory.

IV. HIGHER TRACES

We will now show how the semiclassical trace formula (3.11) generalizes to higher traces $\mathrm{tr}P^N$. For our $2j+1$ dimensional Hilbert space, the knowledge of $(2j+1)^2$ traces suffices in principle to reconstruct the entire spectrum. As a first step we look at the second trace.

A. The second trace

In principle, $\mathrm{tr}P^2$ could be calculated just as $\mathrm{tr}P$ by writing down the exact discrete sums as in (3.2), transforming them to integrals, and doing the integrals by the SPA. The quadratic form in the exponent would then give rise to a 6×6 matrix, and in general, for the N–th trace, to a $3N\times 3N$ matrix. It turns out that the problem simplifies considerably if one integrates over ν once for all in the propagator itself, so that the latter will only depend on actual initial and final coordinates. The matrix in the quadratic form arising from the SPA is then only a $2N\times 2N$ matrix. Let us therefore write an arbitrary matrix element of P as

$$P_{mk;m'k'} = \sum_{l=-\infty}^{\infty}\int d\nu \sum_{\sigma_1,\sigma_2} B(\mu,\nu,\eta;\tau)C(\nu+\eta,\mu'+\eta') \\ \cdot C^*(\nu-\eta,\mu'-\eta')\exp\left(JG(\mu,\eta;\mu',\eta';\nu)\right)\,, \tag{4.1}$$

where again $m=\mu J$ etc. and $C(\mu,\nu)$ is defined in (2.3). The complex action G reads

$$G(\mu,\eta;\mu',\eta';\nu) = R(\mu,\nu;\eta,\tau) + \mathrm{i}\sigma_1 S(\nu+\eta,\mu'+\eta') + \mathrm{i}\sigma_2 S(\nu-\eta,\mu'-\eta') + \mathrm{i}2\pi l\nu\,. \tag{4.2}$$

The saddle–point equation

$$\partial_\nu G = 0 = \partial_\nu R + \mathrm{i}(\sigma_1\partial_\nu S(\nu+\eta,\mu'+\eta') + \sigma_2\partial_\nu S(\nu-\eta,\mu'-\eta') + 2\pi l) \tag{4.3}$$

will in general have complex solutions $\nu=\overline{\nu}(\mu,\eta;\mu',\eta')$, as long as η and η' are not zero and μ and μ' are not connected by a classical trajectory. It may even be possible that several solutions or no solution at all exist. However, note that the saddle-point equation (4.3) simplifies considerably and gives a physically meaningful fixed point if $\sigma_1=-\sigma_2$ and $\eta=\eta'=l=0$ (and that is the situation we encounter for our dissipative integrable kicked top): We then have $\partial_\nu R(\mu,\nu;\eta,\tau)=0$ which is equivalent to the classical dissipative equation of motion $\overline{\nu}=\mu_d^{-1}(\mu)$ and $a=1$.
When the ν integral is done by a SPA the second derivative $\partial_\nu^2 G|_{\nu=\overline{\nu}}\equiv\partial_{\overline{\nu}}^2 G$ comes into play; it can be combined with the other preexponential factors of P. One needs a relation between second derivatives of G which can be obtained by differentiating with respect to μ in the saddle-point equation (4.3), accounting for the μ dependence of $\overline{\nu}$,

$$\partial_{\overline{\nu}}^2 G = -\partial_\mu\partial_{\overline{\nu}} G\frac{1}{\frac{\partial\overline{\nu}}{\partial\mu}} = -\partial_\mu\partial_{\overline{\nu}} R(\mu,\overline{\nu};\eta,\tau)\frac{1}{\frac{\partial\overline{\nu}}{\partial\mu}}\,. \tag{4.4}$$

With the abbreviation $\Psi(\mu,\eta;\mu',\eta')=G(\mu,\eta;\mu',\eta';\overline{\nu}(\mu,\eta;\mu',\eta'))$ for the action at the saddle point we find

$$P_{mk;m'k'} = \sum_{l=-\infty}^{\infty}\sum_{\sigma_1,\sigma_2}\sum_{\overline{\nu}}\sqrt{\frac{\partial\overline{\nu}}{\partial\mu}}\sqrt{\frac{\partial\overline{\nu}}{\partial\mu}\Big|_a} \\ \cdot C(\overline{\nu}+\eta,\mu'+\eta')C^*(\overline{\nu}-\eta,\mu'-\eta')\exp\left(J\Psi(\mu,\eta;\mu',\eta')\right)\,. \tag{4.5}$$

The sum over $\overline{\nu}$ picks up all relevant saddles. To avoid possible confusion we should add that of the two derivatives $\frac{\partial\overline{\nu}}{\partial\mu}$ only the one originating from the semiclassical dissipative propagator is meant at constant a, cf. (2.16), as long as μ and μ' are not connected by the classical dissipative trajectory.

We are now ready to apply the map twice. Let us call R^i and D^i the i-th unitary and dissipative step of the iterated map. The coordinates (μ,η) are transformed in the following way:

$$(\mu_1,\eta_1)\xrightarrow{R^1}(\nu_1,\eta_2)\xrightarrow{D^1}(\mu_2,\eta_2)\xrightarrow{R^2}(\nu_2,\eta_3)\xrightarrow{D^2}(\mu_3,\eta_3)=(\mu_1,\eta_1)\,.$$

The last equation is the periodicity condition for $\mathrm{tr}P^2$. The total action Ψ_2 that appears in the expression for $\mathrm{tr}P^2$ is basically the sum of two actions Φ (cf. (3.4)), one from each repetition of the map. All variables carry an additional index $i=1,2$ counting the iteration of the map (in the case of the σ it is the first index),

$$\begin{aligned}\Psi_2 = {} & R(\mu_1,\overline{\nu}_2;\eta_1) + R(\mu_2,\overline{\nu}_1;\eta_2) + \mathrm{i}\sigma_{21}S(\overline{\nu}_2+\eta_1,\mu_2+\eta_2) \\ & +\mathrm{i}\sigma_{22}S(\overline{\nu}_2-\eta_1,\mu_2-\eta_2) + \mathrm{i}\sigma_{11}S(\overline{\nu}_1+\eta_2,\mu_1+\eta_1) + \mathrm{i}\sigma_{12}S(\overline{\nu}_1-\eta_2,\mu_1-\eta_1) \\ & +\mathrm{i}2\pi((r_1(\mu_1+\eta_1)+l_1(\overline{\nu}_2+\eta_1)+2t_1\eta_1 + r_2(\mu_2+\eta_2)+l_2(\overline{\nu}_1+\eta_2)+2t_2\eta_2)\,.\end{aligned}$$

The last line comes from the Poisson summation over r_i, l_i, and t_i. The μ's and η's are the independent integration variables whose values will now be determined by saddle–point equations.
We assume again that the solution of the SPE's is real and split them into their real and imaginary parts. From the real part of the SPE's with respect to η, $\partial_{\eta_1}\Psi_2 = 0 = \partial_{\eta_2}\Psi_2$, we get $\eta_1 = \eta_2 = 0$. The real part of $\partial_{\mu_1}\Psi_2 = \partial_{\mu_2}\Psi_2 = 0$ leads to the classical dissipative equations of motion at constant ϕ. The imaginary parts of $\partial_{\eta_1}\Psi_2 = 0 = \partial_{\eta_2}\Psi_2$ give

$$(\sigma_{21}-\sigma_{22})\phi_f(\overline{\nu}_2,\mu_2) = (\sigma_{11}-\sigma_{12})\phi_i(\overline{\nu}_1,\mu_1) + (r_1+l_1+2t_1)2\pi \tag{4.6}$$

$$(\sigma_{21}-\sigma_{22})\phi_i(\overline{\nu}_2,\mu_2) = (\sigma_{11}-\sigma_{12})\phi_f(\overline{\nu}_1,\mu_1) + (r_2+l_2+2t_2)2\pi\,. \tag{4.7}$$

These are conditions on the initial and final azimuthal angles for the two classical maps corresponding to the two unitary parts. Let us look at them more closely. Suppose that in any of these equations $\sigma_{i1} = \sigma_{i2}$. We would get $\phi_i = 0$ or $\phi_f = 0$, and we have excluded such cases from our present analysis. If on the other hand we had $(\sigma_{21}-\sigma_{22}) = -(\sigma_{11}-\sigma_{12}) \neq 0$, this would imply $\phi_f(\overline{\nu}_1,\mu_1) = -\phi_i(\overline{\nu}_2,\mu_2)$ (modulo π), which is not possible since during dissipation ϕ is conserved. We therefore conclude that only the two combinations $\sigma_{i1} = -\sigma_{i2} \equiv \sigma = \pm 1$ contribute. Then the SPE's define a period-2 point of the combined map.
If we expand Ψ_2 around a periodic point into a quadratic form, a 4×4 matrix $Q_{4\times 4}$ of second derivatives appears which consists of 2×2 blocks (μ,η), (η,μ), and (η,η) where (μ,η) contains the mixed derivatives $-\partial_{\mu_i}\partial_{\eta_j}\Psi_2$ (and correspondingly for the other blocks):

$$Q_{4\times 4} = \begin{pmatrix} 0 & (\mu,\eta) \\ (\eta,\mu) & (\eta,\eta)\end{pmatrix}\,. \tag{4.8}$$

The (μ,μ) block vanishes. To see this, observe first that its imaginary part gives zero as $\sigma_{i1} = -\sigma_{i2}$. For the real part we have by construction $\partial_{\overline{\nu}_2}\Psi_2 = 0 = \partial_{\overline{\nu}_2}R(\mu_1,\overline{\nu}_2;0)$ *for all* μ_1. But then also $\partial_{\mu_1}R(\mu_1,\overline{\nu}_2;0) = 0$ holds *for all* μ_1 (cf. the general properties of R!). This immediately gives $\partial_{\mu_1}\partial_{\mu_1}\Psi_2 = 0$. A corresponding argument holds for $\partial_{\mu_2}\partial_{\mu_2}\Psi_2$. Furthermore, as $\overline{\nu}_2 = \mu_d^{-1}(\mu_1)$ on the classical trajectory, μ_1 does not even appear in $\partial_{\mu_1}R(\mu_1,\overline{\nu}_2;0)$ and we thus obtain $\partial_{\mu_2}\partial_{\mu_1}R(\mu_1,\overline{\nu}_2;0) = 0$.

Given the structure of the matrix $Q_{4\times 4}$, the (η,η) block is irrelevant, since [28]

$$\det Q_{4\times 4} = (\det(\mu,\eta))^2\,. \tag{4.9}$$

In order to calculate the mixed derivatives in the (μ,η) block, one needs partial derivatives of the $\overline{\nu}$'s with respect to the η's, as we have for example

$$\partial_{\eta_2}\partial_{\mu_1}\Psi_2 = \partial_{\overline{\nu}_2}\partial_{\mu_1}R\frac{\partial\overline{\nu}_2}{\partial\eta_2} + 2\mathrm{i}\sigma\partial_{\mu_1}\partial_{\overline{\nu}_1}S(\overline{\nu}_1,\mu_1)\,; \tag{4.10}$$

they are obtained by totally differentiating $\partial_{\overline{\nu}_i}\Psi = 0$ with respect to the η's and then setting $\eta_1 = \eta_2 = 0$; the one needed for the above example reads

$$\frac{d\overline{\nu}_2}{d\eta_2}\mid_{\eta_1=\eta_2=0} = -2\mathrm{i}\sigma\frac{\partial_{\overline{\nu}_2}\partial_{\mu_2}S(\overline{\nu}_2,\mu_2)}{\partial^2_{\overline{\nu}_2}R(\mu_1,\overline{\nu}_2;0)}\,. \tag{4.11}$$

Upon transforming the remaining second mixed derivative of R in (4.10) into second pure derivatives as in the case of $\mathrm{tr}P$ we can eliminate the derivatives of R entirely. We find for the above example

$$\partial_{\eta_2}\partial_{\mu_1}\Psi_2 = 2\mathrm{i}\sigma\left(\frac{\partial\overline{\nu}_2}{\partial\mu_1}\partial_{\overline{\nu}_2}\partial_{\mu_2}S(\overline{\nu}_2,\mu_2) + \partial_{\mu_1}\partial_{\overline{\nu}_1}S(\overline{\nu}_1,\mu_1)\right)\,. \tag{4.12}$$

In a similar way the remaining three partial derivatives of Ψ_2 can be found. If we abbreviate $S_i = S_i(\overline{\nu}_i,\mu_i)$ ($i = 1,2$, $\overline{\nu}_3 \equiv \overline{\nu}_1$) and $R_i = R_i(\mu_i,\overline{\nu}_{i+1};0)$), $\det Q_{4\times 4}$ is given by

$$\begin{aligned}\det Q_{4\times 4} = 4^2\,[\, & \frac{\partial\overline{\nu}_2}{\partial\mu_1}\left((\partial^2_{\overline{\nu}_2}S_2)(\partial^2_{\mu_2}S_2) - (\partial_{\overline{\nu}_2}\partial_{\mu_2}S_2)^2\right) + \frac{\partial\overline{\nu}_1}{\partial\mu_2}\left((\partial^2_{\overline{\nu}_1}S_1)(\partial^2_{\mu_1}S_1) - (\partial_{\overline{\nu}_1}\partial_{\mu_1}S_1)^2\right) \\ & +\frac{\partial\overline{\nu}_2}{\partial\mu_1}\frac{\partial\overline{\nu}_1}{\partial\mu_2}\left((\partial^2_{\overline{\nu}_1}S_1)(\partial^2_{\overline{\nu}_2}S_2) - (\partial_{\overline{\nu}_1}\partial_{\mu_1}S_1)(\partial_{\overline{\nu}_2}\partial_{\mu_2}S_2)\right) \\ & +(\partial^2_{\mu_1}S_1)(\partial^2_{\mu_2}S_2) - (\partial_{\overline{\nu}_1}\partial_{\mu_1}S_1)(\partial_{\overline{\nu}_2}\partial_{\mu_2}S_2)\,]^2\end{aligned} \tag{4.13}$$

and entails

$$\mathrm{tr}P^2 = 4\sum_{p.p.} \frac{\left|(\partial_{\overline{\nu}_1}\partial_{\mu_1}S_1)(\partial_{\overline{\nu}_2}\partial_{\mu_2}S_2)\frac{\partial\overline{\nu}_1}{\partial\mu_2}\frac{\partial\overline{\nu}_2}{\partial\mu_1}\right|}{\left|\det Q_{4\times 4}\right|^{1/2}} \mathrm{e}^{J(R_1+R_2)} \,. \tag{4.14}$$

The sum is over all periodic points of the map P^2. The various prefactors from the dissipative parts have canceled, up to two factors $\frac{\partial\overline{\nu}_i}{\partial\mu_{i+1}} = \frac{d\mu_d^{-1}}{d\mu_{i+1}} = 1/\frac{d\mu_d(\overline{\nu}_i)}{d\overline{\nu}_i}$. Again, for each periodic point the overall sign of the summand should be determined. It turns out to be positive (for a proof we refer to the discussion in the case of the Nth trace below). The preexponential factor can again be written in terms of elements of the classical monodromy matrices, as can be verified in a straightforward but lengthy calculation. We introduce monodromy matrices $M_d^{(i)}$ and $M_r^{(i)}$ for the dissipative and unitary evolution in the i–th step, respectively, and

$$M^{(i)} = M_d^{(i)} M_r^{(i)} \tag{4.15}$$

for the entire i–th step. The total monodromy matrix M is given by the product $M = M^{(N)} \dots M^{(1)} \equiv \prod_{i=N}^{1} M^{(i)}$. With this convention, the second trace reads

$$\mathrm{tr}P^2 = \sum_{p.p.} \frac{\mathrm{e}^{J(R_1+R_2)}}{\left|\mathrm{tr}M_d^{(2)}M_d^{(1)} - \mathrm{tr}M\right|} \,. \tag{4.16}$$

B. The N–th trace

Our results for the first and the second trace suggest a generalization for the N–th trace,

$$\mathrm{tr}P^N = \sum_{p.p.} \frac{\mathrm{e}^{J\sum_{i=1}^{N} R_i}}{\left|\mathrm{tr}\prod_{i=N}^{1} M_d^{(i)} - \mathrm{tr}M\right|} \,, \tag{4.17}$$

where the sum is now over periodic points of P_{cl}^N. The rest of this section is devoted to the proof of the formula. Starting point is the intermediate semiclassical form (4.5) of the propagator P. We have to consider a sequence of intermediate coordinates of the form

$$\begin{pmatrix}\mu_1\\ \eta_1\end{pmatrix} \xrightarrow{R^1} \begin{pmatrix}\nu_1\\ \eta_2\end{pmatrix} \xrightarrow{D^1} \begin{pmatrix}\mu_2\\ \eta_2\end{pmatrix} \xrightarrow{R^2} \begin{pmatrix}\nu_2\\ \eta_3\end{pmatrix} \xrightarrow{D^2} \begin{pmatrix}\mu_3\\ \eta_3\end{pmatrix} \dots \xrightarrow{R^N} \begin{pmatrix}\nu_N\\ \eta_{N+1}\end{pmatrix} \xrightarrow{D^N} \begin{pmatrix}\mu_{N+1}\\ \eta_{N+1}\end{pmatrix} = \begin{pmatrix}\mu_1\\ \eta_1\end{pmatrix} . \tag{4.18}$$

All variables are periodically continued in their indices ($\nu_{N+i} \equiv \nu_i$, etc.). The total action is given by

$$\Psi_N = \sum_{i=1}^{N} R(\mu_i, \overline{\nu}_{i-1}; \eta_i) + \mathrm{i}\sum_{i=1}^{N} \left(\sigma_{i1} S(\overline{\nu}_i + \eta_{i+1}, \mu_i + \eta_i) + \sigma_{i2} S(\overline{\nu}_i - \eta_{i+1}, \mu_i - \eta_i)\right) \,. \tag{4.19}$$

We have omitted for clarity the term coming from the Poisson summation which is easily seen to lead to integer multiples of 2π in the equality of initial and final angles that we shall find presently. For convenience we introduce a vector notation for the sets of variables, $\boldsymbol{\mu} = \mu_1, \dots, \mu_N$, and similarly for $\overline{\boldsymbol{\nu}}$ and $\boldsymbol{\eta}$. The functions $\overline{\nu}_i = \overline{\nu}_i(\mu_{i+1}, \eta_{i+1}; \mu_i, \eta_i)$ satisfy

$$\partial_{\overline{\nu}_i} \Psi_N(\boldsymbol{\mu}, \overline{\boldsymbol{\nu}}, \boldsymbol{\eta}) = 0 \,. \tag{4.20}$$

Everything goes through for the saddle–point equations as for the first and second trace. Thus, $\sigma_{i1} = -\sigma_{i2} = \sigma = \pm 1$, and the SPE's pick up the periodic points of P_{cl}^N. The $2N \times 2N$ matrix $Q_{2N\times 2N}$ has the same block structure as in the case $N = 2$, that is $\partial_{\mu_i}\partial_{\mu_j}\Psi = 0$ for all i, j. Therefore its determinant depends only on the $N \times N$ (μ, η) block which we will call $B_{N\times N}$ in the following [28]:

$$\det Q_{2N\times 2N} = (-1)^N (\det B_{N\times N})^2 \,. \tag{4.21}$$

The partial derivatives in $B_{N\times N}$ are determined in the same way as for $N = 2$ and contain only derivatives of the imaginary part of Ψ. We find

$$\partial_{\eta_k}\partial_{\mu_l}\Psi_N = 2\mathrm{i}\sigma\Big(\partial_{\overline{\nu}_{l-1}}\partial_{\mu_{l-1}}S_{l-1}\delta_{l-1,k} + \left[\partial^2_{\overline{\nu}_{l-1}}S_{l-1}\frac{\partial\overline{\nu}_{l-1}}{\partial\mu_l} + \partial^2_{\mu_l}S_l\right]\delta_{l,k}$$
$$+\partial_{\overline{\nu}_l}\partial_{\mu_l}S_l\frac{\partial\overline{\nu}_l}{\partial\mu_{l+1}}\delta_{l+1,k}\Big)\,, \tag{4.22}$$

where $\delta_{l,k}$ is the Kronecker–delta.

After the saddle–point integration is done, the expression for the Nth trace reads

$$\mathrm{tr}P^N = 2^N\sum_{p.p.}\left(\prod_{l=1}^{N}\frac{\partial\overline{\nu}_l}{\partial\mu}\big|_{a=1}|C(\overline{\nu}_l,\mu_l)|^2\right)\sqrt{\frac{(2\pi)^{2N}}{J^{2N}|\det B_{N\times N}|^2}}\mathrm{e}^{J\sum_{i=1}^{N}R_i}\,. \tag{4.23}$$

To proceed, one needs to calculate the determinant of $B_{N\times N}$, which is a tridiagonal non–symmetric matrix with additional non-zero elements in the upper right and lower left corners as well. The calculation of such a determinant can be found in the appendix B, where we show that the determinant is the difference between the traces of two different products of 2×2 matrices. Combining equations (4.22), (B14) and (4.23) we are lead to

$$\mathrm{tr}P^N = \sum_{p.p.}\frac{\mathrm{e}^{J\sum_{i=1}^{N}R_i}}{|\mathrm{tr}\prod_{l=N}^{1}M_d^{(l)} - \mathrm{tr}\prod_{l=N}^{1}M_l|}\,. \tag{4.24}$$

The inverted order of the indices at the product indicates that the matrices are ordered from left to right according to decreasing indices. The matrix $M_d^{(l)}$ in the denominator is already the monodromy matrix for the purely dissipative part, $M_d^{(l)} = \mathrm{diag}(\frac{d\mu_d(\overline{\nu}_l)}{d\overline{\nu}_l},1)$. Unfortunately, a corresponding statement does not hold for M_l which is given by

$$M_l = -\frac{1}{\partial_{\nu_l}\partial_{\mu_l}S_l}\begin{pmatrix}\partial^2_{\overline{\nu}_l}S_l + (\partial^2_{\mu_{l+1}}S_{l+1})\frac{d\mu_d(\overline{\nu}_l)}{d\overline{\nu}_l} & -(\partial_{\overline{\nu}_l}\partial_{\mu_l}S_l)^2\\ \frac{d\mu_d(\overline{\nu}_l)}{d\overline{\nu}_l} & 0\end{pmatrix}. \tag{4.25}$$

Indeed, the monodromy matrix $M^{(l)}$ for the entire lth step (cf. (4.15)) has the form

$$M^{(l)} = \frac{1}{\partial_{\nu_l}\partial_{\mu_l}S_l}\begin{pmatrix}-(\partial^2_{\mu_l}S_l)\frac{d\mu_d(\overline{\nu}_l)}{d\overline{\nu}_l} & \frac{d\mu_d(\overline{\nu}_l)}{d\overline{\nu}_l}\\ -(\partial_{\overline{\nu}_l}\partial_{\mu_l}S_l)^2 + (\partial^2_{\overline{\nu}_l}S_l)(\partial^2_{\mu_l}S_l) & -\partial^2_{\overline{\nu}_l}S_l\end{pmatrix} \tag{4.26}$$

when expressed in terms of derivatives of S_l and μ_d. The fact that in M_l both the indices l and $l+1$ appear makes it even impossible to find a similarity transformation independent of l that transforms M_l to $M^{(l)}$. Nevertheless, we will show now that the traces of $\prod_{l=N}^{1}M_l$ and $\prod_{l=N}^{1}M^{(l)}$ are equal. To do so, let us first pull out the common factor $-1/\partial_{\overline{\nu}_l}\partial_{\mu_l}S_l$ from M_l and $M^{(l)}$ by defining $\overline{M}_l = (-\partial_{\overline{\nu}_l}\partial_{\mu_l}S_l)M_l$ and $\overline{M}^{(l)} = (-\partial_{\overline{\nu}_l}\partial_{\mu_l}S_l)M^{(l)}$. Then we separate the parts proportional to $\frac{d\mu_d(\overline{\nu}_l)}{d\overline{\nu}_l}$ from those independent of $\frac{d\mu_d(\overline{\nu}_l)}{d\overline{\nu}_l}$,

$$\overline{M}_l = \frac{d\mu_d(\overline{\nu}_l)}{d\overline{\nu}_l}K_l + L_l,\ \text{and}\ \overline{M}^{(l)} = \frac{d\mu_d(\overline{\nu}_l)}{d\overline{\nu}_l}K^{(l)} + L^{(l)}\,. \tag{4.27}$$

We abbreviate $\alpha_l = \partial^2_{\mu_l}S_l$, $\beta_l = \partial^2_{\overline{\nu}_l}S_l$ and $\gamma_l = \partial_{\overline{\nu}_l}\partial_{\mu_l}S_l$ and find that the matrices on the rhs of the equations in (4.27) have the structure

$$K_l = \begin{pmatrix}\alpha_{l+1} & 0\\ 1 & 0\end{pmatrix},\ L_l = \begin{pmatrix}\beta_l & -\gamma_l^2\\ 0 & 0\end{pmatrix} \tag{4.28}$$

$$K^{(l)} = \begin{pmatrix}\alpha_l & -1\\ 0 & 0\end{pmatrix},\ L^{(l)} = \begin{pmatrix}0 & 0\\ \gamma_l^2 - \alpha_l\beta_l & \beta_l\end{pmatrix}. \tag{4.29}$$

Note that α_l, β_l and γ_l are periodic in the index, i.e. $\alpha_{l+N} = \alpha_l$ etc.. The product of the matrices M_l (or $M^{(l)}$) gives rise to a sum of 2^N terms, each of which can be written as some factors $\frac{d\mu_d}{d\overline{\nu}_l}$ multiplied with a trace over a product of matrices K_l and L_l (or $K^{(l)}$ and $L^{(l)}$, respectively). To complete the proof we must show that the traces over products containing matrices K_l and L_l remain unchanged if we replace all K_l by $K^{(l)}$ and all L_l by $L^{(l)}$. Note that these products are always ordered such that the indices decrease from left to right; and all indices from 1 to N appear.

We will show the equality of the traces by establishing rules for calculating them for all possible combinations of L's and K's. The rules will turn out to be the same for both upper and lower index L's and K's.

Let us express our matrices in terms of spin–$\frac{1}{2}$ operators $S_i = \hat{\sigma}_i/2$ and unity, where $i = x, y, z$ and the $\hat{\sigma}_i$ are the Pauli matrices. In fact only raising and lowering operators $S_\pm = S_x \pm iS_y$, and $\frac{1}{2} \pm S_z$ appear,

$$K_l = S_- + \alpha_{l+1}(\frac{1}{2} + S_z) \tag{4.30}$$

$$L_l = -\gamma_l^2 S_+ + \beta_l(\frac{1}{2} + S_z) \tag{4.31}$$

$$K^{(l)} = -S_+ + \alpha_l(\frac{1}{2} + S_z) \tag{4.32}$$

$$L^{(l)} = (\gamma_l^2 - \alpha_l\beta_l)S_- + \beta_l(\frac{1}{2} - S_z)\,. \tag{4.33}$$

Such a decomposition greatly facilitates the calculation of products of K's and L's due to the simple multiplication rules of the spin operators. The rules can be summarized as

$$S_-^2 = S_+^2 = 0,\ S_\pm S_\mp = (\frac{1}{2} \pm S_z) \tag{4.34}$$

$$(\frac{1}{2} \pm S_z)^n = (\frac{1}{2} \pm S_z) \text{ for all natural } n;\ S_\pm(\frac{1}{2} \pm S_z) = (\frac{1}{2} \pm S_z)S_\mp = 0 \tag{4.35}$$

$$(\frac{1}{2} \pm S_z)S_\pm = S_\pm(\frac{1}{2} \mp S_z) = S_\pm\,. \tag{4.36}$$

Let us start by calculating the traces of products of lower–index matrices.

1. Consider a product of matrices that contains only factors K_l, $\prod_{l=j}^k K_l$, where $j \geq k$ are two arbitrary positive integers. With the above rules we easily find $\prod_{l=j}^k K_l = K_j \prod_{l=j}^{k+1} \alpha_l$ and therefore, $\mathrm{tr} \prod_{l=N}^1 K_l = \prod_{l=N}^1 \alpha_l$.

2. A product containing only factors L_l leads to $\prod_{l=j}^k L_l = L_k \prod_{l=j}^{k+1} \beta_l$ and thus to $\mathrm{tr} \prod_{l=N}^1 L_l = \prod_{l=N}^1 \beta_l$.

3. A product L_kK_j gives $L_kK_j = (\alpha_{j+1}\beta_k - \gamma_k^2)(\frac{1}{2} + S_z)$. With this result we can calculate a product which contains L's and K's,

$$L_k \dots L_jK_{j-1} \dots K_i = (\prod_{l=k}^{j+1} \beta_l)(\alpha_j\beta_j - \gamma_j^2)(\prod_{l=j-1}^{i+1} \alpha_l)(\frac{1}{2} + S_z)\,, \tag{4.37}$$

   where $k \geq j \geq i$. Since $(\frac{1}{2} \pm S_z)^n = (\frac{1}{2} \pm S_z)$ for all integer n, we can easily generalize the above equation to products containing several L and K blocks. We thus obtain the following rule for the trace of an arbitrary product starting with L_N and ending with K_1:

   - Replace all L_l by β_l with the exception of the last L_l in each L block.
   - Replace all K_l by α_l with the exception of the last K_l in each K block.
   - Each interface between an L block and a K block (in this order, i.e. a term L_jK_{j-1}!) gives rise to a factor $\alpha_j\beta_j - \gamma_j^2$.

   We will call these three rules the *standard rules*.

4. The next case to be considered is a product starting with L_N and ending with L_1, with an arbitrary number of L_l and K_l in between. We can apply (4.37) to all blocks with the exception of the last L block, to which we apply directly the rule for a pure L block. Besides the previous replacement of K_l by α_l and L_l by β_l, one obtains a factor $(\frac{1}{2} + S_z)L_1 = L_1$. Since $\mathrm{tr}L_1 = \beta_1$ the standard rules have to be modified only in as much as $\mathrm{tr}L_N \dots LK \dots K \dots L_1$ gives rise to an additional factor β_1.

5. Consider now a product starting with K_N and ending with L_1. Due to the cyclic property of the trace we can bring the last L block to the left (e.g. $\mathrm{tr}K_N \dots K_jL_{j-1} \dots L_1 = \mathrm{tr}L_{j-1} \dots L_1K_N \dots K_j$), thus creating an interface L_1K_N; and the product of matrices now starts with an L and terminates with a K matrix. Thus, the L and K blocks and their interfaces can be replaced according to the standard rules. Due to the cyclic properties of K_l and L_l, also the interface L_1K_N is of the type L_jK_{j-1} with $j = 1$ and is therefore to be replaced by $\alpha_1\beta_1 - \gamma_1^2$. In summary, the standard rules apply, but we get an additional factor $\alpha_1\beta_1 - \gamma_1^2$.

6. The last situation is a product starting with K_N and ending with K_1. Here it is useful to treat the first block $K_N \dots K_j$ separately according to the rule for a pure K block and the subsequent sequence $L_{j-1} \dots K_1$ according to (4.37). One therefore encounters a factor $\mathrm{tr} K_N(\frac{1}{2} + S_z) = \alpha_1$. So again the standard rules apply, but we get an additional factor α_1.

Let us now consider the corresponding expressions for the matrices with upper indices resulting from the monodromy matrices.

1. A product containing only factors $L^{(l)}$ gives an expression with the same structure as a product of factors L_l, $\prod_{l=j}^{k} L^{(l)} = L^{(k)} \prod_{l=j}^{k+1} \beta_l$. It follows immediately that $\mathrm{tr} \prod_{l=N}^{1} L^{(l)} = \prod_{l=N}^{1} \beta_l$.

2. A product consisting only of factors $K^{(l)}$ behaves differently from its counterpart with lower indices, for the last and not the first $K^{(l)}$ survives, $\prod_{l=j}^{k} K^{(l)} = K^{(k)} \prod_{l=j}^{k+1} \alpha_l$. However, since, $\mathrm{tr} K^{(1)} = \alpha_1$ we still have $\mathrm{tr} \prod_{l=N}^{1} K^{(l)} = \prod_{l=N}^{1} \alpha_l$.

3. A product of $K^{(j)}$ and $L^{(l)}$ is now simpler with $K^{(j)}$ on the left, $K^{(j)} L^{(l)} = (\alpha_l \beta_l - \gamma_l^2)(\frac{1}{2} + S_z) - \beta_l S_\pm$. We can combine the rules for the pure K blocks, the pure L blocks and the KL–interface to derive the product

$$K^{(k)} \dots K^{(j)} L^{(j-1)} \dots L^{(i)} = \left(\prod_{l=k}^{j+1} \alpha_l \right) \left(\prod_{l=j-1}^{i+1} \beta_l \right) \left[(\alpha_i \beta_i - \gamma_i^2)(\frac{1}{2} + S_z) - \beta_i S_+ \right] . \tag{4.38}$$

   If we have several such KL blocks, each of them gives rise to a factor corresponding to the one in angular brackets on the rhs of the expression. But note that $\left[(\alpha_i \beta_i - \gamma_i^2)(\frac{1}{2} + S_z) - \beta_i S_+ \right] \left[(\alpha_l \beta_l - \gamma_l^2)(\frac{1}{2} + S_z) - \beta_l S_+ \right] = (\alpha_i \beta_i - \gamma_i^2) \left[(\alpha_l \beta_l - \gamma_l^2)(\frac{1}{2} + S_z) - \beta_l S_+ \right]$, i.e. from all KL blocks only the last angular bracket survives whereas all others are replaced by factors $(\alpha_i \beta_i - \gamma_i^2)$ where the index is always the last index in the L blocks. We thus recover the rule of the replacement of LK interfaces by factors $(\alpha_i \beta_i - \gamma_i^2)$. In an arbitrary product starting with $K^{(N)}$ and ending in $L^{(1)}$ the last angular bracket leads to the factor $(\alpha_1 \beta_1 - \gamma_1^2)$; and all K blocks, L blocks, and LK interfaces have to be replaced by the standard rules! Thus, we get just the same expression as for a product starting with K_N and ending with L_1.

4. In a product starting with $K^{(N)}$ and ending with $K^{(1)}$ it is useful to treat the last K block separately. If l is the last index of an L in the product, one is lead to a factor $\mathrm{tr}((\alpha_l \beta_l - \gamma_l^2)(\frac{1}{2} + S_z) - \beta_l S_+) K^{(1)} = \mathrm{tr}(\alpha_l \beta_l - \gamma_l^2) K^{(1)} = (\alpha_l \beta_l - \gamma_l^2)\alpha_1$. Thus the standard rules apply, but we get an additional factor α_1, in agreement with the result for the corresponding lower–index expression.

5. In $\mathrm{tr} L^{(N)} \dots L^{(j)} K^{(j-1)} \dots KL \dots L^{(1)}$ we treat the first L block separately. One easily sees then that again the standard rules apply up to an additional factor β_1, as was the case for the corresponding expression with lower indices.

6. The last product to be considered is one that starts with $L^{(N)}$ and ends with $K^{(1)}$. We use the cyclic properties of the trace to shift the first L block to the right. Then (4.38) applies, and we see that the last $L^{(j)}$ term on the right gives the same factor $(\alpha_j \beta_j - \gamma_j^2)$ as did the $L_j K_{j-1}$ interface in the corresponding expression with lower indices. The standard rules apply without restriction.

So we have shown that an arbitrary ordered product of factors $K^{(l)}$ and $L^{(l)}$ (l decreasing from N to 1 from left to right) has the same trace as the corresponding product in which all $K^{(l)}$ are replaced by K_l and all $L^{(l)}$ by L_l. This proves that $\mathrm{tr} \prod_{l=N}^{1} M_l = \mathrm{tr} \prod_{l=N}^{1} M^{(l)}$.

The last thing to consider is the sign of each saddle point contribution for arbitrary N. We will show now that it is always possible to choose all minors of $Q_{2N \times 2N}$ real and positive. Therefore the sign of each saddle point contribution in (4.17) is positive.

Observe first of all that without regularization all minors D_l of $Q_{2N \times 2N}$ with the exception of the determinant $\det Q_{2N \times 2N}$ itself are zero. This is obvious for $l = 1 \dots N$, since there the corresponding matrix is part of the upper left zero block of $\det Q_{2N \times 2N}$. For $l = N + m > N$ note that D_l contains a $N \times N$ upper left block which is zero, and a $N \times m$ (μ, η) block in the upper right corner. Upon expanding D_l after the first row one encounters subdeterminants with a $(N-1) \times N$ upper left zero block and a $(N-1) \times (m-1)$ upper right (μ, η) block. Both blocks together have $N-1$ rows in each of which only the $m-1$ elements on the right can be different from zero. Therefore the $N-1$ first rows are always linearly dependent, unless $m = N$, the case that amounts to the full determinant. Thus, all minors D_l with $1 \leq l \leq 2N-1$ are zero.

Suppose now that we add to Ψ_N a small quadratic term that vanishes and has a maximum at the saddle point $(\boldsymbol{\mu},\boldsymbol{\eta}) = (\boldsymbol{\mu}_{sp},0)$, i.e. a function $-\epsilon\sum_{i=1}^{N}((\mu_i-\mu_{sp_i})^2+\eta_i^2)$ with infinitesimal $\epsilon>0$. If the original integral is convergent, the small addition will not change the value of the integral in the limit $\epsilon\to 0$, but allows us to determine the phase of all minors D_l. The matrices $\mathbf{D}_l$ corresponding to D_l get all replaced by $\mathbf{D}'_l=\mathbf{D}_l+\epsilon\mathbf{1}_l$, where $\mathbf{1}_l$ is the unit matrix in l dimensions. For $1\le l\le N$ we are immediately lead to $D_l=\epsilon^l>0$. For $N+1\le l\le 2N-1$ we expand D'_l in powers of ϵ and get $D'_l=D_l+\epsilon\,\mathrm{tr}\mathbf{D}_l+\mathcal{O}(\epsilon^2)=\epsilon\,\mathrm{tr}\mathbf{D}_l+\mathcal{O}(\epsilon^2)$. To determine the traces $\mathrm{tr}\mathbf{D}_l$ we need the second derivatives in the (η,η) block of $Q_{2N\times 2N}$. They are given by

$$\partial_{\eta_k}\partial_{\eta_l}\Psi_N=\left(\partial^2_{\eta_k}R(\mu_k,\overline{\nu}_{k+1};\eta_k)+4\frac{(\partial_{\overline{\nu}_k}\partial_{\mu_k}S_k)^2}{\partial^2_{\overline{\nu}_k}R(\mu_{k-1},\overline{\nu}_k;\eta_{k-1})}+4\frac{(\partial^2_{\overline{\nu}_{k-1}}S_{k-1})^2}{\partial^2_{\overline{\nu}_{k-1}}R(\mu_{k-2},\overline{\nu}_{k-1};\eta_{k-2})}\right)\delta_{k,l}$$
$$+4\frac{(\partial^2_{\overline{\nu}_{k-1}}S_{k-1})(\partial_{\overline{\nu}_{k-1}}\partial_{\mu_{k-1}}S_{k-1})}{\partial^2_{\overline{\nu}_{k-1}}R(\mu_{k-2},\overline{\nu}_{k-1};\eta_{k-2})}\delta_{k-1,l}+4\frac{(\partial_{\overline{\nu}_k}\partial_{\mu_k}S_k)(\partial^2_{\overline{\nu}_k}S_k)}{\partial^2_{\overline{\nu}_k}R(\mu_{k-1},\overline{\nu}_k;\eta_{k-1})}\delta_{k+1,l}\,. \tag{4.39}$$

Since $\partial^2_{\eta_k}R(\mu_k,\overline{\nu}_{k+1};\eta_k)<0$ and $\partial^2_{\overline{\nu}_k}R(\mu_{k-1},\overline{\nu}_k;\eta_{k-1})<0$ for all k at the saddle point, the diagonal elements of the (η,η) block of $Q_{2N\times 2N}$ are all real and positive definite (remember that we pulled out a minus sign in the definition of $Q_{2N\times 2N}$ in terms of second derivatives). Therefore $\mathrm{tr}\mathbf{D}_l$ is real and larger than zero for all l. Since also $\det Q_{2N\times 2N}$ is real and positive, the sign of each saddle–point contribution is now determined to be positive (cf.(A3)). The proof of our trace formula (4.17) for the Nth trace is herewith complete. It is worth mentioning that the comments made about the decoherence built in in the formula (3.11) for $\mathrm{tr}P$, the limit $\tau\to 0$, and the generality of the formula apply to (4.17) for $\mathrm{tr}P^N$as well .

V. EVALUATION FOR AN INTEGRABLE DISSIPATIVE KICKED TOP

We now propose to evaluate explicitly $\mathrm{tr}P^N$ for our integrable dissipative kicked top. To do so, we need the periodic points of the classical map P_{cl} and their stabilities.

A. Periodic points

Let us start by determining the fixed points of P_{cl}. It is most convenient to work with Cartesian coordinates, $x=J_x/J$, $y=J_y/J$, and $z=J_z/J=\mu$ (with $x^2+y^2+z^2=1$). The coordinates x'_N, y'_N and z'_N immediately after the Nth rotation are related to those immediately before the Nth rotation (x_N, y_N and z_N) by

$$x'_N=x_N\cos\beta+z_N\sin\beta \tag{5.1}$$
$$y'_N=y_N \tag{5.2}$$
$$z'_N=-x_N\sin\beta+z_N\cos\beta\,. \tag{5.3}$$

The dissipation takes these intermediate coordinates to those immediately before the $(N+1)$st rotation,

$$x_{N+1}=\frac{x'_N}{\cosh\tau-z'_N\sinh\tau} \tag{5.4}$$
$$y_{N+1}=\frac{y'_N}{\cosh\tau-z'_N\sinh\tau} \tag{5.5}$$
$$z_{N+1}=\frac{-\sinh\tau+z'_N\cosh\tau}{\cosh\tau-z'_N\sinh\tau}\,. \tag{5.6}$$

For $\tau=0$, the dissipative part becomes the identical map and the fixed points are those of the rotation, thus $\mathbf{x}_{N+1}=\mathbf{x}_N=(0,\pm1,0)$ for all β.
For $\tau\neq 0$, the equation for the y component demands $y_N=0$ or $\cosh\tau-z'_N\sinh\tau=1$. Let us consider the latter equation first. It leads to the fixed points

$$x_N=-\cot\frac{\beta}{2}\tanh\frac{\tau}{2},\quad y_N=\pm\sqrt{\frac{1-\cos\beta-2\tanh^2\frac{\tau}{2}}{1-\cos\beta}},\quad z_N=-\tanh\frac{\tau}{2}\,. \tag{5.7}$$

They can exist with real y_N only if $\sinh\frac{\tau}{2} \leq |\tan\frac{\beta}{2}|$. If we compare (5.7) with $\mathbf{x}_{t+1} = \mathbf{x}_t = (0, \pm 1, 0)$ at $\tau = 0$, we see that the pair of fixed points shifts downwards in z and towards the $y = 0$ plane as τ increases. When $\tau = \tau_c$ given by $\sinh\frac{\tau_c}{2} = |\tan\frac{\beta}{2}|$, the two fixed points coincide in the plane $y = 0$; a bifurcation takes place and one has to consider $y_N = 0$.
This case is treated most easily in polar coordinates, since $y_N = 0$ means $\phi = 0$ and $\mathbf{x}$ depends only on the polar angle Θ. The rotation becomes trivial, $\Theta' = \Theta_N + \beta$, where Θ should be allowed to run through $-\pi ... \pi$ in order to avoid $\phi = \pi$. The overdamped-pendulum trajectory (2.9) implies

$$\tan\frac{\Theta_{N+1}}{2} = \mathrm{e}^{\tau}\tan\frac{\Theta'}{2} = \mathrm{e}^{\tau}\tan\frac{\Theta_N + \beta}{2} \tag{5.8}$$

throughout the interval $\Theta = -\pi ... \pi$. With the substitution $u_N = \tan\frac{\Theta_N}{2}$ the periodic-point condition can be written as a quadratic equation,

$$\tan\frac{\beta}{2}\, u_N^2 + u_N(1 - \mathrm{e}^{\tau}) + \tan\frac{\beta}{2}\mathrm{e}^{\tau} = 0\,. \tag{5.9}$$

The two solutions

$$u_{N_{1,2}} = \frac{\mathrm{e}^{\tau/2}}{\tan(\beta/2)}\left(-\sinh\frac{\tau}{2} \pm \sqrt{\sinh^2\frac{\tau}{2} - \tan^2\frac{\beta}{2}}\right) \tag{5.10}$$

are real iff $\sinh\frac{\tau}{2} \geq |\tan\frac{\beta}{2}|$, i.e.$\tau > \tau_c$. Transforming u_N back to μ, we get

$$\mu = \pm\cos\frac{\beta}{2}\sqrt{1 - (\coth\frac{\tau}{2}\,\sin\frac{\beta}{2})^2} - \coth\frac{\tau}{2}\,\sin^2\frac{\beta}{2}\,. \tag{5.11}$$

For $\tau \to \infty$, the lower one of the two fixed points moves to the south pole $\mu = -1$ for all values of β. The upper one moves to $\mu = -\cos\beta$, such that the intermediate point immediately after the rotation is at the north pole, $\mu = 1$. The final momentum ν is obtained from (5.11) by reversing the sign in front of the last $\coth\frac{\tau}{2}$.
The boundary between classically allowed and forbidden rotation is given by $\Theta_N + \Theta' = \beta$ (see section II A), or

$$\sin^2\beta - \mu^2 - \nu^2 + 2\mu\nu\cos\beta = 0\,. \tag{5.12}$$

One easily checks that the two fixed points given by (5.11) have the remarkable property that they lie exactly on this boundary, for all values of $\tau > \tau_c$ and β, as long as the fixed points exist. Unfortunately, in this case the semiclassical expression (2.4) for the unitary propagator ceases to be valid. To avoid this problem we will restrict ourselves to $\tau < \tau_c$.
We show in appendix C that P_{cl}^N has the same and only the same fixed points as P_{cl}. Therefore the restriction $\tau < \tau_c$ is sufficient as well for all higher traces. We also remark that $N\beta$ should not be an integer multiple of 2π, otherwise already the semiclassical approximation of the unitary part breaks down [18].

In FIG.1 we show phase–space portraits. The northern and southern hemisphere of the spherical phase space spanned by the variables μ and ϕ were projected on the equatorial plane $\mu = 0$. The two symmetry related fixed points are clearly visible for $\tau < \tau_c$ as *elliptic fixed points*, whereas for $\tau > \tau_c$ they become a *pair of a point attractor and a point repeller.*

B. Stability matrices

The only non–trivial element of the monodromy matrix M_d for the dissipative motion is given by $\frac{\partial\mu(\nu,\phi)}{\partial\nu} = \frac{d\mu_d}{d\nu} = \frac{1-\mu^2}{1-\nu^2}$, whereas $\frac{\partial\phi_d(\nu,\phi)}{\partial\phi} = 1$ and the two other elements are zero. For the fixed point at $\tau < \tau_c$, $\nu = -\mu$, even this non–trivial element equals one, and thus $M_d = \mathbf{1}$. In the vicinity of its two fixed points our dissipative map is actually area–preserving.
The elements of M_r can be found from differentiating the equations of motion for the rotation written in μ, ϕ and $\mu' = \nu, \phi'$,

$$\sqrt{1-\nu^2}\sin\phi' = \sqrt{1-\mu^2}\sin\phi \tag{5.13}$$

$$\nu = \mu\cos\beta - \sqrt{1-\mu^2}\cos\phi\sin\beta\,. \tag{5.14}$$

We then obtain at $\nu = -\mu$

$$\frac{\partial \phi'}{\partial \phi} \mid_{\mu} = 1 - \frac{\sin^2 \beta}{1+\cos\beta} \sin^2 \phi \tag{5.15}$$

$$\frac{\partial \mu'}{\partial \mu} \mid \phi = \cos\beta + \frac{\sin^2 \beta}{1+\cos\beta} \cos^2 \phi \,, \tag{5.16}$$

and thus at the fixed points (5.7),

$$\mathrm{tr}M = \mathrm{tr}M_1 = \frac{2}{1-\tanh^2 \frac{\tau}{2}} (\tanh^2 \frac{\tau}{2} + \cos\beta) \,. \tag{5.17}$$

If we insert this and $\mathrm{tr}M_d = 2$ in (3.11), we obtain the final result for $\tau < \tau_c$,

$$\mathrm{tr}P = \frac{1-\tanh^2 \frac{\tau}{2}}{1-\cos\beta-2\tanh^2 \frac{\tau}{2}} \,. \tag{5.18}$$

A factor 2 has canceled since two orbits contribute with equal weight. Clearly, $\mathrm{tr}P$ diverges at the bifurcation. Since M_1 is unitary for $\frac{\partial \mu_d}{\partial \nu} = 1$, its eigenvalues can be written as $\lambda_{1/2} = \mathrm{e}^{\pm \mathrm{i}\delta}$. We have therefore $\mathrm{tr}M_1 = 2\cos\delta$ and $\mathrm{tr}M_1^N = \cos(N\delta) = \cos(N \arccos \frac{\mathrm{tr}M_1^N}{2})$, or

$$\mathrm{tr}P^N = \frac{1}{1-\cos(N \arccos(\frac{\tanh^2 \frac{\tau}{2}+\cos\beta}{1-\tanh^2 \frac{\tau}{2}}))} \,. \tag{5.19}$$

C. Comparison with numerics

In FIG.2 we have plotted the first trace as semiclassically approximated by (5.18) together with the numerically obtained quantum counterpart for different values of j. The agreement is good for $\tau \gtrsim 1/J$ and τ not too close to the critical τ_c, where the bifurcation takes place and the semiclassics diverges; the agreement improves with increasing j. The rather erratic behavior of $\mathrm{tr}P$ as a function of J when $\tau \lesssim 1/J$ is a signature of the importance of quantum effects in the dissipative part.

In FIG.3 we compare the semiclassical and quantum mechanical ($j = 20$) result for $\mathrm{tr}P^N$ as a function of N. The agreement is rather good for the first about 10 traces. Both results show oscillations, and the semiclassical result reproduces perfectly their period and phase over the entire N range. However, the exact traces decay exponentially as $\mathrm{tr}P^N \stackrel{N\to\infty}{\longrightarrow} 1$, whereas the semiclassical approximation (5.19) shows undamped oscillations. Note that all eigenvalues λ_i of P are bound by $|\lambda_i| \leq 1$ due to stability reasons. In general one has even $|\lambda_i| < 1$ with the exception of one eigenvalue $\lambda_1 = 1$ which corresponds to a stationary state. Since $\mathrm{tr}P^N = \sum_{i=1}^{(2j+1)^2} \lambda_i^N$ the exact traces must indeed decay exponentially towards 1.

We have studied the decay of the exact traces numerically in more detail. It is an exponential decay of the form $\mathrm{tr}P^N \simeq 1 + const.\mathrm{e}^{\frac{-f(\tau,\beta)N}{J}}$, where $f(\tau,\beta)$ is some function of the parameters. Thus, the effect of the dissipation is very small and vanishes for fixed finite N in the classical limit. Its onset should be visible one order beyond our present semiclassical approximation. Note that the absence of classical periodic orbits with phase–space contraction alone is not enough to expect a spectrum corresponding to a unitary map (i.e. eigenvalues on the unit circle). This is due to the fact that beyond the classical information entering via the periodic points our trace formula always contains a dissipative effect on the quantum mechanical level, namely the complete destruction of coherence manifest in the single sum over periodic points and the cancellation of the actions from the unitary part.

In order to arrive at the precision at which the exponential decay should be seen, it would be necessary to obtain at least the purely dissipative propagator D to accuracy $\mathcal{O}(1/J)$. So far our semiclassical calculations of $\mathrm{tr}P$ have been restricted by the requirement $1/J \ll \tau \ll J$ which marked the limit of validity of the semiclassical approximation of D. For the calculation of $\mathrm{tr}P^N$ one should have correspondingly $N/J \ll N\tau \ll J$.

In order to reconstruct the whole spectrum one would need traces from $N = 1$ to $N = (2J)^2$, such that τ could only be less than $1/J$ to fulfill the last condition. But this conflicts with $\tau \gg 1/J$. We conclude therefore that without pushing semiclassics one order further, it will never be possible to calculate a whole spectrum. However,

this might be a too ambitious goal anyway: Even for moderate dissipation ($\tau \simeq 2$ for example), most eigenvalues are exponentially small and can not be reconstructed numerically even if one knew the exact traces! We tested this by calculating traces from exact quantum mechanical eigenvalues. From the traces one can get the coefficients of the characteristic polynomial via Newton's formulae and therefore in principle back the eigenvalues. But the polynomial becomes exponentially small for its argument close to zero, which makes it virtually impossible to accurately determine its roots. For large values of J one faces moreover the problem that the higher traces ($N \sim J^2$) are all equal to unity up to tiny corrections of order $\mathcal{O}(\exp(-J))$ and therefore carry no more information.
On the other hand, it is quite likely that in typical experimental situations only the slowest decaying eigenmodes and corresponding eigenvalues can be measured. Future investigations will have to show, in how far these can be obtained from a limited number of traces.

VI. CONCLUSION

We have established a semiclassical periodic–orbit theory for a dissipative quantum map with an area–nonpreserving classical limit. We have shown that traces of arbitrary integer powers of the propagator of the density matrix can be written as a sum over classical periodic points. Our results generalize the well known formula due to Tabor for area–preserving maps and are in the spirit of Gutzwiller's trace formula. Both the exponent and preexponential factor in Tabor's formula are modified. A "diagonal approximation" [29,30] which amounts to neglecting interferences between different orbits in suitably averaged quantities arises automatically for our dissipative case: Interferences between different orbits are destroyed by the dissipation. Therefore the exponent in the trace formula is purely real. Only the action from the dissipative part of the map remains. Even though only classical information is needed to evaluate our trace formula, the quantum mechanical effect of the destruction of coherence by dissipation is automatically built in. In the preexponent the total monodromy matrix, but also the monodromy matrix of the dissipative parts alone enter. The Maslov index is always zero.

Our theory allows to quantify the effects of the dissipation on the spectrum of the propagator even when dissipation is so strong that it not only leads to dephasing but also to a modification of periodic orbits. Our example of a dissipative integrable kicked top showed, however, that even though the classical map may be area–nonpreserving globally, it might have periodic orbits close to which it *is* area–preserving. If only such orbits exist, the only influence of the dissipation on the spectrum is via the destruction of interferences between different periodic orbits.

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APPENDIX A: SADDLE–POINT METHOD FOR A COMPLEX FUNCTION OF SEVERAL VARIABLES

Let $\mathbf{x}$ represent M real variables $x_1 \dots x_M$, and let $F(\mathbf{x})$ and $G(\mathbf{x})$ be complex valued functions of $\mathbf{x}$ with $\Re F \leq 0$ in the volume V in $\mathcal{R}^M$ over which we are going to integrate. Suppose that V contains a single non–degenerate stationary point $\mathbf{x}_0$ with $\partial_{x_i} F(\mathbf{x}_0) = 0$, $i = 1 \dots M$. Let us denote by $Q_{M\times M}$ the matrix of the negative second derivatives, $(Q_{M\times M})_{ik} = -\partial_{x_i}\partial_{x_k} F(\mathbf{x})$ taken at $\mathbf{x} = \mathbf{x}_0$. The condition of non–degeneracy means $\det Q_{M\times M} \neq 0$. We then have for $J \to \infty$ [27]

$$\int_V d^M\mathbf{x} \mathrm{e}^{JF(\mathbf{x})} G(\mathbf{x}) = G(\mathbf{x}_0)\sqrt{\frac{(2\pi)^M}{J^M|\det Q_{M\times M}|}}\mathrm{e}^{JF(\mathbf{x}_0)-\frac{\mathrm{i}}{2}\mathrm{Ind}Q_{M\times M}}(1+\mathcal{O}(1/J))\,. \tag{A1}$$

Here $\mathrm{Ind}Q_{M\times M}$ is the index of the complex quadratic form χ of M real variables,

$$\chi = \sum_{i,j=1}^{M} (Q_{M\times M})_{ij} x_i x_j\,. \tag{A2}$$

The index is defined via the minors $D_k = \det||(Q_{M\times M})_{ij}||$, $1 \leq i,j \leq k$ of $Q_{M\times M}$ as

$$\mathrm{Ind}Q_{M\times M} = \sum_{k=1}^{M} \arg\rho_k,\ -\pi < \arg\rho_k \le \pi\,, \tag{A3}$$

$$\rho_1 = D_1 = (Q_{M\times M})_{11},\ \rho_k = \frac{D_k}{D_{k-1}},\quad k = 2\ldots M\,. \tag{A4}$$

The restriction $-\pi \le \arg\rho_k \le \pi$ on the phases fixes uniquely the overall phase of the saddle point contribution. Without this restriction $\mathrm{Ind}Q_{M\times M}$ would only be defined up to multiples of 2π, and this would lead to an overall phase ambiguity corresponding to the choice of sign for the square root in (A1).

In the case of a $D_k = 0$ we add a term $-\epsilon(x_i - x_{0_i})^2$ to the function $F(\mathbf{x})$. Such an addition does not change the convergence of the integral (in the limit of $J \to \infty$), nor its value for $\epsilon \to 0$. However, such a small term may bring the D_k away from zero and therefore allow us to determine its phase.

Formula (A1) has the structure familiar from the SPA of a complex function of a single variable. The term $-\frac{\mathrm{i}}{2}\mathrm{Ind}Q_{M\times M}$ leads to a phase analogous to Maslov's $\pm\pi/4$, but that phase can now take on any value between $-\frac{\pi}{2}\cdots\frac{\pi}{2}$.

APPENDIX B: THE DETERMINANT OF A TRIDIAGONAL, PERIODICALLY CONTINUED MATRIX

The $N \times N$ matrix whose determinant we want to calculate has the structure

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & \ldots & 0 & a_{1N} \\ a_{21} & a_{22} & a_{23} & 0 & \ldots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & & & 0 \\ \vdots & & & & & & \vdots \\ a_{N1} & 0 & \ldots\ldots\ldots & & & a_{N,N-1} & a_{NN} \end{pmatrix} \tag{B1}$$

We first consider the simpler problem where $a_{1N} = a_{N1} = 0$; the corresponding matrix will be called A_0. The determinant of A_0 can be expanded along the diagonal by using the block decomposition rule for determinants [28]. If A is decomposed into smaller matrices W $(n\times n)$, X $(n\times m)$, Y $(m\times n)$, and Z $(m\times m$, where $n+m=N)$ and if the inverse of W exists, we have

$$\det\begin{pmatrix} W & X \\ Y & Z \end{pmatrix} = \det W \det(Z - YW^{-1}X)\,. \tag{B2}$$

We apply the rule to $\det A_0$ by choosing $W = a_{11}$. Then $Z \equiv A^{(1)}$ is the matrix in which the first column and row are missing. In the matrix $YW^{-1}X$ only one element is different from zero, namely $(YW^{-1}X)_{ik} = \delta_{i,2}\delta_{k,2}\frac{a_{21}a_{12}}{a_{11}}$. Note that we keep the numbering of indices of the original matrix A. Thus, in the determinant on the rhs of (B2), which is now only $(N-1)\times(N-1)$, the indices run from 2 to N. The new matrix whose determinant has now to be calculated differs from the corresponding block in A just in the upper left element a_{22}, and is tridiagonal again. The procedure may therefore be iterated. In each step the dimension of the matrix is decreased by one; the only matrix element changed is always the one in the upper left corner. It gets replaced by the original one in A_0 minus the product of the two secondary diagonal elements flanking it divided by the upper left element of one step earlier. One obtains therefore a product of renormalized diagonal elements,

$$\det A_0 = \prod_{j=1}^{N} c_j \tag{B3}$$

with $c_1 = a_{11}$ and the iteration law

$$c_j = a_{jj} - \frac{a_{j,j-1}a_{j-1,j}}{c_{j-1}}\,. \tag{B4}$$

If we evaluate the product starting from the last term, we find that all c_j's with the exception of the one with the smallest index cancel, such that

$$\prod_{j=n}^{N} c_j = c_n f_n + g_n\,. \tag{B5}$$

The functions f_n and g_n depend only on the matrix elements, not on the c_j's. In the notation of (B5) $\det A_0$ is given by $\det A_0 = c_1 f_1 + g_1$, and the final condition for f_n and g_n reads $f_N = 1$, $g_N = 0$. A recursion relation for f_n and g_n is easily found by equating $\prod_{j=n-1}^{N} c_j = c_{n-1}(c_n f_n + g_n) = c_{n-1}(a_{nn} f_n + g_n) - a_{n,n-1}a_{n-1,n} f_n = c_{n-1} f_{n-1} + g_{n-1}$. We obtain a linear relation which we write with the help of the matrix

$$\mathbf{Q}_n = \begin{pmatrix} a_{nn} & 1 \\ -a_{n,n-1}a_{n-1,n} & 0 \end{pmatrix} \tag{B6}$$

as

$$\begin{pmatrix} f_{n-1} \\ g_{n-1} \end{pmatrix} = \mathbf{Q}_n \begin{pmatrix} f_n \\ g_n \end{pmatrix}. \tag{B7}$$

The linear combination $c_1 f_1 + g_1$ at our interest therefore reads

$$(c_1, 1)\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} = (a_{11}, 1)\left(\prod_{j=2}^{N} \mathbf{Q}_j\right)\begin{pmatrix} f_N \\ g_N \end{pmatrix}.$$

The product can be extended down to $j = 1$ by noting that the vector $(a_{11}, 1)$ is just the first row of the matrix in the product that would carry the index 1. In the second row the product of elements $-a_{1,0}a_{0,1}$ appears. Its value is at this stage completely arbitrary. We define it by periodically continuing the indices, i.e. $a_{1,0} \equiv a_{1,N}$ and $a_{0,1} \equiv a_{N,1}$. We then arrive at the result for the determinant of the tridiagonal matrix A_0,

$$\det A_0 = (1, 0)\left(\prod_{j=1}^{N} \mathbf{Q}_j\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{B8}$$

Consider now the case where $a_{1,N}a_{N,1} \neq 0$. The problem of calculating $\det A$ can be reduced to the problem of determinants of purely tri–diagonal matrices by expanding $\det A$ after the first row and the resulting submatrices after the first column. One eliminates in this way the rows and columns with the disturbing additional corner elements. A matrix $A^{(1,N)}$ appears, which is defined as the matrix A from which both the first and Nth rows and columns have been taken out. The reader will easily verify that

$$\det A = \det A_0 - a_{1N}a_{N1} \det A^{(1,N)} + (-1)^{N+1}\left(\prod_{j=1}^{N} a_{j,j-1} + \prod_{j=1}^{N} a_{j-1,j}\right). \tag{B9}$$

For $a_{1N}a_{N1} = 0$ the expression reduces to $\det A_0$ as it should. It will be useful to rewrite the sum of the two products as trace of a diagonal 2×2 matrix which is the product of matrices $\mathbf{R}_j \equiv \mathrm{diag}(a_{j,j-1}, a_{j-1,j})$. Let us apply now our result for tri–diagonal determinants (B8). We obtain

$$\det A = (1, 0)\left[\left(\prod_{j=1}^{N} \mathbf{Q}_j\right) - a_{1,N}a_{N,1}\left(\prod_{j=2}^{N-1} \mathbf{Q}_j\right)\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-1)^{N+1}\mathrm{tr}\prod_{j=1}^{N} \mathbf{R}_j\,. \tag{B10}$$

Now observe that

$$(0, 1)\mathbf{Q}_1 = -a_{1,N}a_{N,1}(1, 0) \text{ and} \tag{B11}$$

$$\mathbf{Q}_N\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{B12}$$

Thus, while the first term in (B10) gives the upper left matrix element of the product of matrices, the second one gives the lower right. We therefore have

$$\det A = \mathrm{tr}\prod_{j=1}^{N} Q_j + (-1)^{N+1}\mathrm{tr}\prod_{j=1}^{N} \mathbf{R}_j\,. \tag{B13}$$

The order of the products is such that they start with the matrices with index 1 on the left. Given the ordering of the monodromy matrices that appears in the main text it is useful to inverse the order. This can easily be done by taking the traces of the transposed product, which leads to the final result

$$\det A = \mathrm{tr} \prod_{j=N}^{1} \begin{pmatrix} a_{jj} & -a_{j,j-1}a_{j-1,j} \\ 1 & 0 \end{pmatrix} + (-1)^{N+1} \mathrm{tr} \prod_{j=N}^{1} \begin{pmatrix} a_{j,j-1} & 0 \\ 0 & a_{j-1,j} \end{pmatrix} . \tag{B14}$$

Do not apply the formula for $N = 2$, since we assumed *additional* elements in the corners. For $N = 2$ the corners collapse with the secondary diagonal and are therefore over counted in (B14). The lowest dimension for which the formula works is $N = 3$.

APPENDIX C: FIXED POINTS OF P_{CL}^N

Let us first consider the case $y_n \neq 0$. Equating y_{N+1} with y_1 in (5.5), one is lead immediately to $\prod_{i=1}^{N}(\cosh\tau - \sinh\tau z_i') = 1$. Since the same denominator also appears in x_{N+1} and z_{N+1} when expressed as functions of x_1 and z_1, the remaining map $(x_1, z_1) \to (x_{N+1}, z_{N+1})$ is in fact *linear!* Foreseeing this we can build it up by iterating a linear map $\mathbf{A}$ which acts on the two dimensional vector $\zeta_l \equiv (x_l, z_l)$ as

$$\zeta_{l+1} = \mathbf{A}\zeta_l + \mathbf{b} \, . \tag{C1}$$

The matrix $\mathbf{A}$ and the vector $\mathbf{b}$ are explicitly given by

$$\mathbf{A} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\cosh\tau\sin\beta & \cosh\tau\cos\beta \end{pmatrix}, \; \mathbf{b} = \begin{pmatrix} 0 \\ -\sinh\tau \end{pmatrix} . \tag{C2}$$

The condition for the fixed points of P_{cl} reads $(\mathbf{1} - \mathbf{A})\zeta_1 = \mathbf{b}$. The determinant of $(\mathbf{1} - \mathbf{A})$ is $(1 - \cos\beta)(1 + \cosh\tau)$ and therefore non–zero as long as there is any rotation at all. The equation is solved easily and gives back the fixed point at $z_1 = -\tanh\frac{\tau}{2}$, but additionally we now see directly that it is unique in x_1, z_1, as the map is linear and the determinant non–zero. The y component is obtained from normalization, $y_1 = \pm\sqrt{1 - x_1^2 - z_1^2}$. These are the two basic symmetry related fixed points of P_{cl}.
The fixed points of P_{cl}^2 follow from $\zeta_1 = \zeta_3 = \mathbf{A}\zeta_2 + \mathbf{b} = \mathbf{A}^2\zeta_1 + (\mathbf{1} + \mathbf{A})\mathbf{b}$, i.e. $\zeta_1 = (\mathbf{1} - \mathbf{A}^2)^{-1}(\mathbf{1} + \mathbf{A})\mathbf{b} = (\mathbf{1} - \mathbf{A})^{-1}\mathbf{b}$. One easily verifies by complete induction that the formula for the Nth iteration is

$$(\mathbf{1} - \mathbf{A}^N)\zeta_1 = (\sum_{i=1}^{N} \mathbf{A}^i)\mathbf{b} = \left(\frac{1 - A^N}{1 - A}\right)\mathbf{b} \tag{C3}$$

and therefore again just $(\mathbf{1} - \mathbf{A})\zeta_1 = \mathbf{b}$, the equation for the fixed point of P_{cl}.

For $\phi = 0$ the situation is similar: One easily verifies that $u_{N+2} = u_N$ in equation (5.8) leads exactly to the same equation (5.9) as did $u_{N+1} = u_N$, and the same is true for all higher iterations. This completes our prove that P_{cl}^N has the same and only the same fixed points as P_{cl}.

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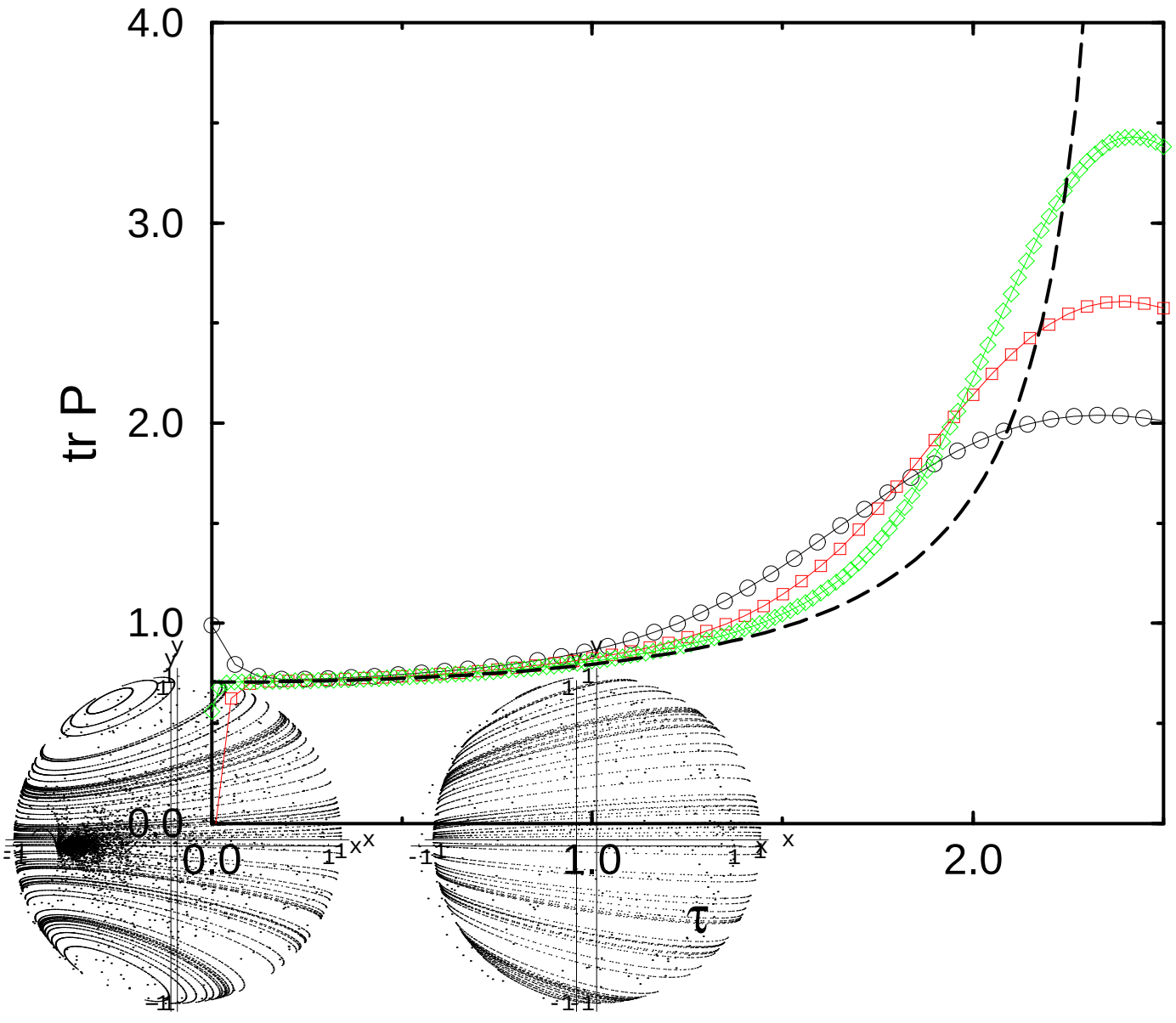
FIG. 1. Phase–space portraits of the classical map of the dissipative integrable kicked top ($\beta = 1.2$). We projected the northern and the southern hemisphere ($\mu > 0$ and $\mu < 0$, respectively) of the spherical classical phase space on the equatorial plane $\mu = 0$; (a) $0.6 = \tau < \tau_c \simeq 1.28$, the two fixed points are elliptic fixed points; (b) $1.3 = \tau > \tau_c \simeq 1.28$, the fixed points form a point attractor/repeller pair. The repeller in the northern hemisphere can be made visible by iterating the inverse map.

FIG. 2.

Comparison of semiclassical (dashed line) and quantum mechanical result for $\mathrm{tr}P$ as a function of the damping τ at $\beta = 2.0$. The semiclassical result (5.18) diverges at $\tau = \tau_c \simeq 2.45$, and the quantum mechanical trace follows this divergence the further the higher the value of J (circles $j = 10$, squares $j = 20$, diamonds $j = 40$). For $\tau \lesssim 1/J$ the problem is purely quantum mechanical and our semiclassical treatment seizes to be valid.

FIG. 3.

Comparison of semiclassical and quantum mechanical result for $\mathrm{tr}P^N$ ($\beta = 0.5$, $\tau = 0.2$) as a function of N. FIG.(a) for $j = 20$ shows that the agreement is fairly good for about the first 10 traces and the quantum mechanical oscillations are well reproduced over the entire N range. However, the exact traces decay, whereas the semiclassical approximation (5.19) only shows the oscillations (same symbols as in FIG.1). In (b) the exponential decay of the exact traces is studied numerically for different values of j. It is of the form $\mathrm{tr}P^N \simeq 1 + const. \exp(\frac{-N}{9.5J})$ for $\beta = 0.5$ and $\tau = 0.2$.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3.0
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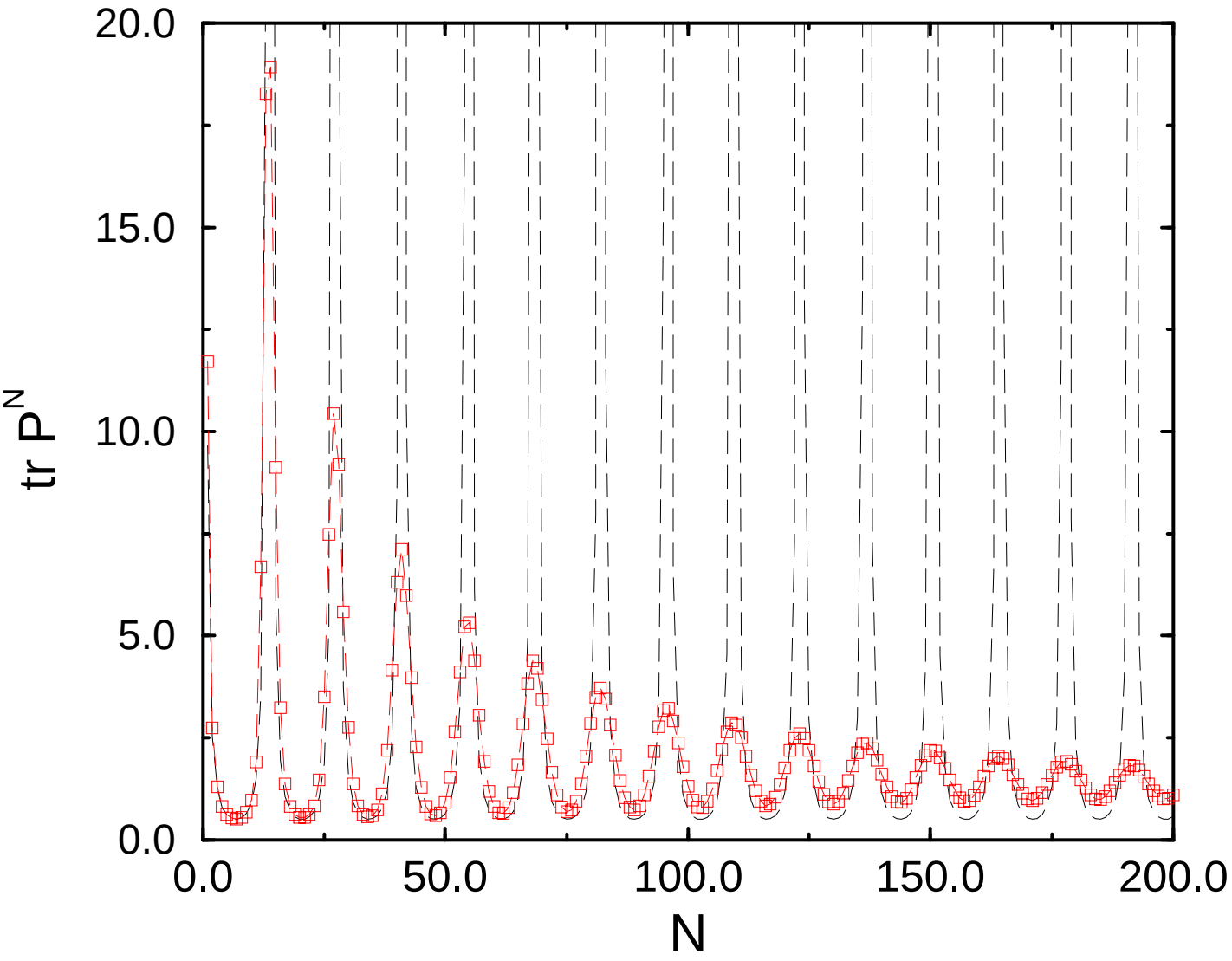
20.0
15.0
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5.0
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tr P^N
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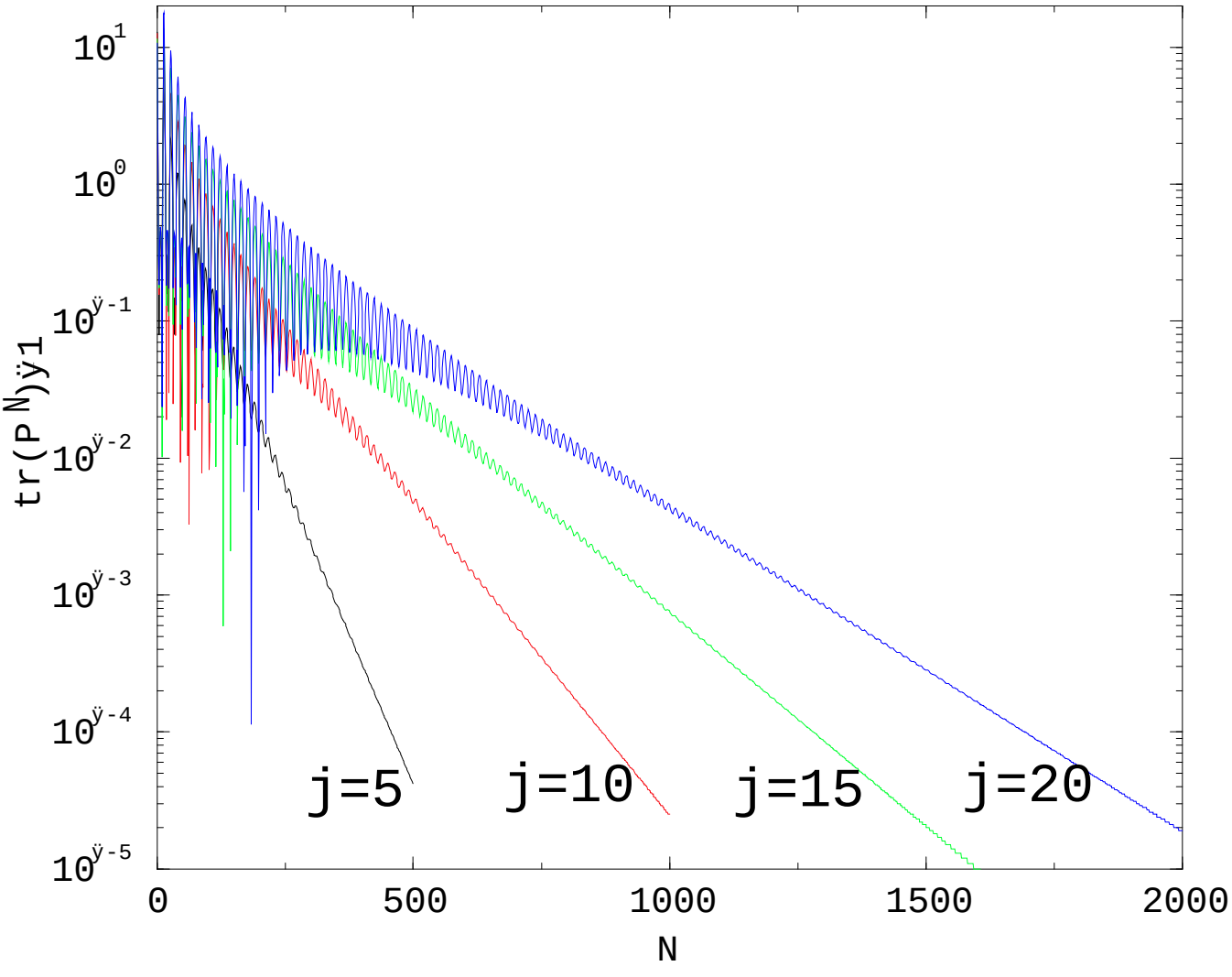
50.0
100.0
150.0
200.0
N


10^1
10^0
10^ÿ-1
10^ÿ-2
10^ÿ-3
10^ÿ-4
10^ÿ-5
tr(P^N)ÿ1
0
500
1000
1500
2000
N
j=5
j=10
j=15
j=20